Quantum effects and quantum paradoxes
Exercise sheet 8

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1. Harmonic oscillator as pointer $(4 + 4 + 3 + 4)$

The pointer of a measurement device is modelled in terms of an harmonic oscillator. The observable which is measured is $A_S = |0\rangle\langle 0|$. The relevant Hamiltonian reads

$$H = g(t) A_S p_Z,$$

where p_Z denotes the momentum operator of the harmonic oscillator which describes the pointer. In terms of the ladder operators the momentum reads $p_Z = i\sqrt{\frac{\hbar m\omega}{2}}(a_Z^\dagger - a_Z)$. We assume that the pointer is initially in the ground state. Moreover, it is $g(t) = \alpha$ for $0 \leq t \leq \tau$ and $g(t) = 0$ otherwise. Solve the exercise in the occupation number representation.

(a) At first, we are interested in which of the matrix elements $\langle kj| \mathrm{e}^{-iH\tau/\hbar} |\ell 0\rangle$ are non-zero. For this purpose, calculate the matrix elements for $k \neq 0$ and $l \neq 0$, where the state $|kj\rangle = |k\rangle_S \otimes |j\rangle_Z$ describes the system S and the pointer Z. Are the matrix elements for $k = l = 0$ non-zero?

(b) Determine $p_{00} = \langle 00| \mathrm{e}^{-iH\tau/\hbar} |00\rangle$ and show that $p_{00} \to 0$ for $\alpha \to \infty$.

(c) How can the measurement result (0 or 1) for A_S be obtained?

(d) Specify the (reduced) state of the system after the measurement, depending on the measurement result.

2. Quantum Zeno effect $(3 + 3 + 3 + 6)$

A two dimensional quantum system has the energy eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ with the corresponding eigenvalues E_1 and E_2. Assume that $E_2 > E_1$. The states are also characterised by parity, which is represented by an operator P that acts on the energy eigenstates as $P|\psi_1\rangle = |\psi_2\rangle$ and $P|\psi_2\rangle = |\psi_1\rangle$.

(a) Find the eigenstates of the parity operator in terms of $|\psi_1\rangle$ and $|\psi_2\rangle$.

(b) Assuming that the system is initially in a positive-parity eigenstate, find the state of the system at any later time $t > 0$.

(c) At a particular time T a parity measurement is made on the system. What is the probability of finding the system with positive parity?

(d) Imagine that instead of a single measurement at time T you make a series of N parity measurements at the times Δt, $2\Delta t$, and so on, up to $N\Delta t = T$. Assuming that N is very large and $\Delta t \ll \frac{\hbar}{E_2 - E_1}$, what is the probability of finding the system with positive parity at time T? Compare this probability with the probability of finding the system in the positive parity state with a single measurement at $t = T$ (that is, your answer to part (c)). This 'freezing' of the system in the initial state for a repeated series of measurements has been called the 'quantum Zeno effect'. *Hint: You may find the series expansion*

$$\left(1 - \frac{x^2}{n^2}\right)^n \xrightarrow{n \gg 1} e^{-\frac{x^2}{n}}$$

useful.