Quantum effects and quantum paradoxes
Exercise sheet 7

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1. Purification $(5+2+8)$

In the lecture the partial trace has been introduced. One may ask whether the partial trace can be inverted. To be precise, the question is whether every mixed state ϱ_A of a system A can be seen as a reduced state of a composite system AB, i.e. $\varrho_A = \mathrm{tr}_B\left(|\psi\rangle\langle\psi|_{AB}\right)$. The state $|\psi\rangle_{AB}$ is called purification of ϱ_A. Every density operator can be decomposed in its eigenstates, i.e. $\rho_A = \sum_i p_i\, |i\rangle\langle i|_A$. Then, a purification of ρ_A is given by

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i}\, |i\rangle_A \otimes |i\rangle_B\,,$$

where the states $|i\rangle_B$ are from an orthonormal basis of $\mathcal{H}_B$.

(a) Show that the state $|\psi\rangle_{AB}$ is indeed a purification of ρ_A, i.e. $\varrho_A = \mathrm{tr}_B\left(|\psi\rangle\langle\psi|_{AB}\right)$ holds.

(b) What is the minimal dimension the Hilbert space $\mathcal{H}_B$ has to have such that a purification $|\psi\rangle_{AB}$ exists?

(c) The purification is not unique. Show that two purifications $|\psi\rangle_{AB}$ and $|\phi\rangle_{AB}$ of a state ϱ_A are connected by a unitary transformation which acts in the Hilbert space $\mathcal{H}_B$, i.e. there exists a unitary U_B such that

$$|\psi\rangle_{AB} = (\mathbb{1}_A \otimes U_B)\, |\phi\rangle_{AB}\,.$$

Hint: You may use the relation that if for same matrices A, B it holds $AA^\dagger = BB^\dagger$ then there exists a unitary transformation U such that $A = BU$.

2. Heuristic model of a fluorescence measurement $(5+5+5)$

An atom with four internal states $|0\rangle$, $|1\rangle$, $|2\rangle$, and $|e\rangle$ is initially in the state

$$|\psi\rangle = \alpha_0\, |0\rangle + \alpha_1\, |1\rangle + \alpha_2\, |2\rangle\,.$$

The exited state $|e\rangle$ decays almost instantly into the ground state $|0\rangle$. In this process, a photon is emitted. Thus, the 'pointer' of the measurement is the emitted photon. We distinguish between no photon $|\gamma_0\rangle$ and at least one emitted photon $|\gamma_1\rangle$.

The model works as follows: At first the state of the atom evolves for time τ according to the Hamiltonian $H_D = \eta(|0\rangle\langle e| + |e\rangle\langle 0|)$. This results in a unitary transformation $|0\rangle \leftrightarrow |e\rangle$. Afterwards, the excited state decays according to the (non-unitary) transformation $R = |0\rangle\langle e|_S \otimes |\gamma_1\rangle\langle\gamma_0| + \sum_{i=0}^{2} |i\rangle\langle i| \otimes |\gamma_0\rangle\langle\gamma_0|$.

(a) Calculate the state of the composite system (atom and pointer) after the measurement.

(b) Determine the state of the atom after the measurement for the two cases that one photon or no photon is detected.

(c) Compare your result with the generalized map for a measurement

$$\Phi\colon \rho \mapsto \sqrt{P}\rho\sqrt{P}.$$

(See also exercise 2 on sheet 6.)