Quantum effects and quantum paradoxes
Exercise sheet 6

Lecture: PD. Dr. M. Kleinmann
Exercises: Jan Bönsel

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1. Mach-Zehnder interferometer with phase $(5+5+5)$
In this exercise, we consider the Mach-Zehnder interferometer in Fig. 1. In path c, the photons pick up a phase shift $e^{i\varphi}$. Moreover, the interferometer consists of $50:50$ beam splitters, such that the transformations of beam splitter B_1 can be described as

$$U_{B_1}|a\rangle = (i|c\rangle + |d\rangle)/\sqrt{2},$$
$$U_{B_1}|b\rangle = (|c\rangle + i|d\rangle)/\sqrt{2}.$$

The transformation of B_2 follows analogously.

(a) A single photon is irradiated in mode a. What is the probability to detect the photon at the detector D_1?

We now want to consider the case of two photons in the interferometer. For this purpose, we have to take into account that photons are bosons. Thus, the states have to be symmetric under particle exchange, i.e. in case the states of two photons are switched the composite state does not change. It is useful to work in the occupation number representation. For the modes a, b, the relevant states $|n_a, n_b\rangle$ for two photons read

$$|2_a, 0_b\rangle = |a\rangle_1 \otimes |a\rangle_2\,,$$
$$|1_a, 1_b\rangle = (|a\rangle_1 \otimes |b\rangle_2 + |b\rangle_1 \otimes |a\rangle_2)/\sqrt{2},$$
$$|0_a, 2_b\rangle = |b\rangle_1 \otimes |b\rangle_2\,,$$

where the right hand side is written in terms of the product basis of the two photons, e.g. in the state $|a\rangle_1 \otimes |b\rangle_2$ the first photon is in mode a whereas the second photon is in mode b. The relevant states $|n_c, n_d\rangle$ and $|n_e, n_f\rangle$ are defined accordingly.

(b) Find the transformation of the beam splitter B_1 for two photons in terms of the states $|n_a, n_b\rangle$ and $|n_c, n_d\rangle$. On these lines, write the transformation of B_2 in terms of the states $|n_c, n_d\rangle$ and $|n_e, n_f\rangle$. *Hint: As the photons do not interact, the beam splitters act in the product basis of two photons as* $U_B^{(2)} = U_B \otimes U_B$.

(c) Two photons are irradiated such that the input state is $|1_a, 1_b\rangle$. What is the probability to observe both photons or no photon at detector D_1? Now, we assume that the phase φ is unknown and has to be measured. Is the approach with one photon in (a) or the approach with two photons in (c) more sensitive with respect to the phase φ.

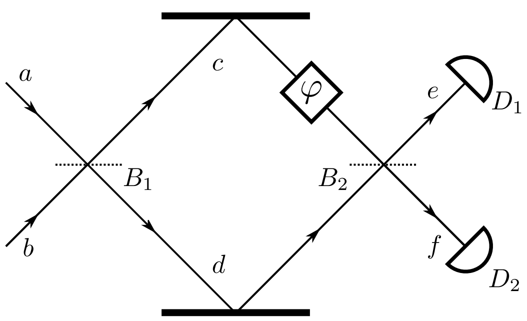

Figure 1: Mach-Zehnder interferometer with phase shift φ.

2. Sorkin's observation $(7 + 8)$

For a projective measurement $B = (\Gamma_1, \Gamma_2, \dots, \Gamma_n)$, the map ϕ_α is defined as

$$\phi_\alpha \colon \rho \mapsto \left(\sum_{i\in\alpha} \Gamma_i\right) \rho \left(\sum_{i\in\alpha} \Gamma_i\right),$$

where $\alpha \subseteq \{1, 2, \dots, n\}$. The interferences η_α of order $|\alpha| = k$ are defined recursively by

$$\phi_\alpha = \sum_{\beta\subseteq\alpha} \eta_\beta, \quad \text{or rather} \quad \eta_\alpha = \phi_\alpha - \sum_{\beta\subsetneq\alpha} \eta_\beta.$$

Sorkin observed that $\eta_\alpha = 0$ for $k > 2$. For the general case, the transformation reads

$$\phi_\alpha \colon \rho \mapsto \sqrt{\sum_{i\in\alpha} X_i}\, \rho \sqrt{\sum_{i\in\alpha} X_i},$$

where $\sum_{i\in\alpha} X_i \geq 0$. For $A \geq 0$, $\sqrt{A}$ is the uniquely defined operator, which fulfils $\sqrt{A}\sqrt{A} = A$ and $\sqrt{A} \geq 0$.

(a) Show that both definitions of ϕ_α are equivalent in case $(X_1, X_2, \dots, X_n)$ are projectors which add up to $\mathbb{1}$.

(b) We consider the example for $n = 3$ with $X_1 = a\,|0\rangle\langle 0|$, $X_2 = b\,|1\rangle\langle 1|$ and $X_3 = \mathbb{1} - X_1 - X_2$ where $0 \leq a \leq 1$ and $0 \leq b \leq 1$. What is the result for $\eta_{\{1,2,3\}}$? When does $\eta_{\{1,2,3\}} = 0$ hold?