Quantum effects and quantum paradoxes
Exercise sheet 5

Lecture: PD. Dr. M. Kleinmann — Due: Thursday, 25.11.2021
Exercises: Jan Bönsel

---

1. Partial trace $(4 + 5 + 6)$
The reduced density operator ρ_A of a bipartite state ρ_{AB} can be calculated as the partial trace

$$\rho_A = \mathrm{tr}_B(\rho_{AB}) = \sum_{i,j,k} |i\rangle\langle j| \langle i,k| \rho_{AB} |j,k\rangle .$$

(a) Show that $\rho_A = \mathrm{tr}_B(\rho_A \otimes \rho_B)$.

(b) The reduced states of a tripartite system can be defined analogously in terms of the partial trace. For example the reduced density operator of the joint subsystem A and B reads $\rho_{AB} = \mathrm{tr}_C(\rho_{ABC})$. Calculate ρ_{AB} for the following tripartite states:

$$\begin{aligned} |\mathrm{GHZ}\rangle_{ABC} &= (|000\rangle + |111\rangle)/\sqrt{3} \quad \text{and} \\ |W\rangle_{ABC} &= (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3} \end{aligned}$$

There are multiple ways, $\mathrm{tr}_C(|W\rangle\langle W|)$ can be decomposed into a mixture of pure states. Argue that any of these decompositions contains at least one entangled bipartite state.

(c) Prove that the definition of the partial trace is independent of the chosen orthonormal bases of A and B.

2. Measurements $(4 + 3 + 4 + 4)$
A measurement $(\Pi_1, \dots, \Pi_n)$ on a system in state ρ with outcome k induces the canonical transformation

$$\rho \mapsto (\Pi_k \rho \Pi_k)/\,\mathrm{tr}(\rho \Pi_k),$$

where Π_k is the projector. In case the outcome is not known after the measurement $(\Pi_1, \dots, \Pi_n)$ (e.g. the result is not read out or is lost), the transformation reads

$$\rho \mapsto \sum_k \Pi_k \rho \Pi_k.$$

(a) The observable $A = |0\rangle\langle 0| - |1\rangle\langle 1| - |2\rangle\langle 2|$ is measured on a system in state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle$, but the outcome is not recorded. Which successive measurements can be performed which are not disturbed by the measurement of A?

(b) The singlet state $|\psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is prepared and party B measures either σ_x or σ_z. For both cases, calculate the reduced state of subsystem A. Once for the situation the measurement outcome is known and once for the case it is unknown.

(c) The state of a system A is mapped to a state of a joint system AB by the transformation

$$\alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle \mapsto \alpha |00\rangle_{AB} + \beta |11\rangle_{AB} + \gamma |21\rangle_{AB} .$$

Is the transformation unitary? For the state after the transformation, calculate the reduced state ρ_A. To what extent does this transformation implements the observable from part (a)?

(d) For $k = 1, 2, \dots, n$, the observables $B_k = \mathbb{1} - 2|\eta_k\rangle\langle\eta_k|$ are measured sequentially on a system in state $|0\rangle$, where $|\eta_k\rangle = \cos[\pi k/(2n)]\,|0\rangle + \sin[\pi k/(2n)]\,|1\rangle$. Coincidentally, all measurements yield the outcome -1. Determine the state of the system after the n measurements. Use numerics to calculate the probability that all outcomes are -1 for all $n \in \{1, \dots, 100\}$ measurements.