Quantum effects and quantum paradoxes
Exercise sheet 4

Lecture: PD. Dr. M. Kleinmann Due: Thursday, 18.11.2021
Exercises: Jan Bönsel

---

1. Interaction-free measurement II $(10 = 5 + 5)$
In this exercise, we will consider the setup in Fig. 1. By parametric down-conversion an entangled photon pair is created in the $LiIO_3$ crystal. Thus, in case a photon is observed at the detector T, we can deduce with certainty that there is exactly one photon in the interferometer. Solve the exercise in the occupation number representation, i.e. by the four states of the from $|n_a, n_b, n_c, n_d\rangle$ with $n_a, n_b, n_c, n_d \in \{0, 1\}$ and $n_a + n_b + n_c + n_d = 1$, where a, b, c, d denote the respective path in Fig. 1. The state $|0\rangle$ corresponds to no photon in the respective path. Note that not all four states are orthogonal. The beam splitter has the reflection coefficient R.

(a) Show that in case the movable mirror is not in the path of the photon, the photon exits the interferometer by the same path it entered, i.e. the detector *Dark* never clicks.

(b) Now, the movable mirror is placed in the path. Calculate the efficiency $\eta = \frac{P_{Dark}}{P_{Dark}+P_{Obj}}$, where P_{Obj} and P_{Dark} denote the probabilities to detect the photon at the corresponding detector. The experiment is repeated multiple times. How can the efficiency be determined from the number of times the photon is detected at T, *Dark* and *Obj*.

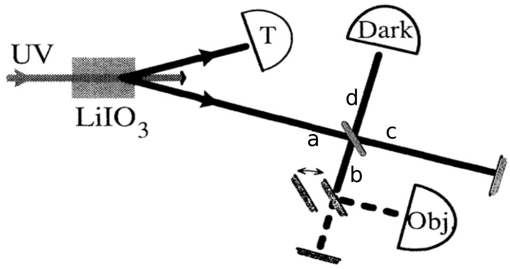

Figure 1: Experimental setup used by Kwiat u. a, Phys. Rev. Lett. **74**, 4763 (1995).

2. Mixed states $(10 = 3 + 2 + 2 + 3)$
A mixed state can be described be a density operator ρ. A density operator ρ has the properties: (i) ρ is positive semi-definite ($\rho \geq 0$) and (ii) $\mathrm{tr}(\rho) = 1$. Show the following statements:

(a) $\mathrm{tr}\big(\rho^2\big) = 1$ if and only if ρ describes a pure state, i.e. $\mathrm{rank}(\rho) = 1$.

(b) There exists exactly one density operator, which is simultaneously invariant under all unitary transformations. Find this specific density operator.

(c) A mixture of two pure states is pure if and only if the two pure states are equal (up to a global phase).

(d) We consider a genuinely mixed state ρ, i.e. $\mathrm{rank}(\rho) > 1$. There are infinitely many possibilities to represent ρ as a mixture of pure states, i.e.

$$\rho = \sum p_k \, |\phi_k\rangle\langle\phi_k|$$

with $p_k > 0$, $\sum_k p_k = 1$, and $|\langle\phi_k|\phi_l\rangle| = 1$ only for $k = l$.

3. Quantum correlations $(10 = 5 + 5)$

We have already encountered the entangled state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. In case Alice and Bob measure σ_z on the first and second qubit respectively, we know that in case Alice obtains the result $+1$, Bob will measure the outcome -1.

(a) Now we suppose that the system is in the state $|\psi_3\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$. Moreover, Alice measures σ_z and Bob measures σ_x What is the probability that Bob measures the result -1? We now assume that Alice has measured $+1$ on her qubit first. Do Bob's chances to measure -1 change? Repeat the calculation for the state $|\psi_2\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. What is the difference?

(b) Let us consider the two qubit mixed state

$$\rho = \frac{1}{8}\mathbf{1} + \frac{1}{2}\left|\psi^-\right\rangle\left\langle\psi^-\right|,$$

where $\mathbf{1}$ denotes the 4×4 unit matrix. Alice and Bob measure both σ_z. Calculate the probability that Bob measures -1. Does the probability change if Alice measures $+1$ on her qubit first?