Quantum effects and quantum paradoxes
Exercise sheet 3

Lecture: PD. Dr. M. Kleinmann — Due: Thursday, 11.11.2021
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1. Tsirelson's bound $(10 = 5 + 5)$

We consider the four observables A_1, A_2 and B_1, B_2, which act on qudits, i.e. in a d-dimensional Hilbert space. Each of the observables has the two possible measurement outcomes ± 1. The Bell operator is defined as

$$\mathcal{B} = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2.$$

Thereby the CHSH inequality reads $\langle \mathcal{B} \rangle \leq 2$. However, you have seen in the lecture that quantum mechanics allow states which violate the CHSH inequality. The maximal value of the expectation value $\langle \mathcal{B} \rangle$ is known as Tsirelson's bound.

(a) Show that the square of the operator $\mathcal{B}$ takes the form $\mathcal{B}^2 = 4\mathbf{1} - [A_1, A_2] \otimes [B_1, B_2]$.

(b) Determine the maximal value $\max_{|\psi\rangle} \langle \mathcal{B}^2 \rangle$. From this result deduce Tsirelson's bound with the help of the relation $\langle \mathcal{B} \rangle^2 \leq \langle \mathcal{B}^2 \rangle$.

2. CHSH II (10)

Let us again consider the CHSH operator $\mathcal{B}$. In this exercise we deal with qubits and choose the observables $A_1 = B_1 = \sigma_x$ and $A_2 = B_2 = \sigma_y$. Is there still a state for which the maximal value $\langle \mathcal{B} \rangle = 2\sqrt{2}$ is reached?

3. Interaction-free measurement $(10 = 4 + 3 + 3)$

The experimental set-up in Fig. 1 depicts a Mach-Zehnder interferometer. The interferometer consists of two input modes E_1 and E_2, two beam splitters $B_{1/2}$, two mirrors and two detectors $D_{1/2}$. The beam splitters are characterized by the reflection coefficients $R_{1/2}$. In case a photon is reflected by a beam splitter, the state picks up the phase i. The phase shift due to the mirrors can be neglected as it constitutes a global phase. A photon in the input mode E_1 is described by the state $|0\rangle$, whereas $|1\rangle$ corresponds to a photon in the input mode E_2. The upper (lower) path is described by the state $|2\rangle$ ($|3\rangle$) and the detectors D_1 (D_2) are described by $|4\rangle$ ($|5\rangle$).

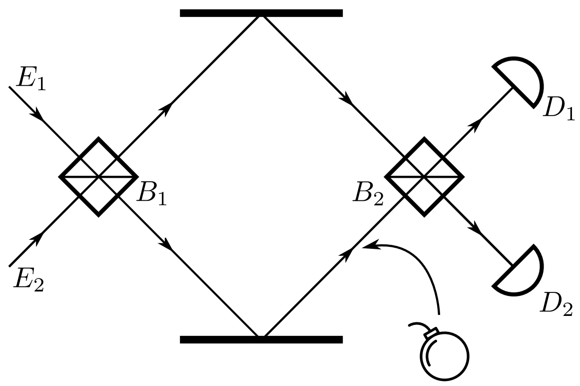

Figure 1: Mach-Zehnder interferometer

The beam splitter B_1 connects the modes $|2\rangle, |3\rangle$ to the input modes $|0\rangle, |1\rangle$ by the unitary transformation

$$\begin{pmatrix} |2\rangle \\ |3\rangle \end{pmatrix} = \begin{pmatrix} i\sqrt{R_1} & \sqrt{1-R_1} \\ \sqrt{1-R_1} & i\sqrt{R_1} \end{pmatrix} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}.$$

An analogous relation holds for beam splitter B_2.
In this exercise, we assume that the incoming photon is always in mode E_2.

(a) Assume that the value of R_1 is fixed. First, we consider the interferometer with no object placed in the paths of the photon. Determine R_2 such that the photon is always detected at D_1 and never at D_2.

(b) Then an object (bomb) is placed in the lower path. Calculate the efficiency

$$\eta = \frac{P_{\text{det}}}{P_{\text{det}} + P_{\text{abs}}}$$

as a function o R_1. P_{det} is the probability that an interaction-free measurement has taken place, whereas P_{abs} denotes the probability that the photon is absorbed by the bomb and the bomb explodes.

(c) The experiment is repeated if the photon is detected at the detector D_1. What is the asymptotic efficiency?