Quantum effects and quantum paradoxes
Exercise sheet 1

Lecture: PD. Dr. M. Kleinmann — Due: Thursday, 28.10.2021
Exercises: Jan Bönsel

Information: The first two exercises are presence tasks and do not have to be handed in. Rather, they will be discussed in the first tutorial on Monday 25.10.2021.

1. Matrices, eigenvalues and eigenvectors (presence task)
We define

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

(a) Calculate the eigenvalues and eigenvectors of the matrices σ_i.

(b) Write the matrices σ_i in their spectral decomposition.

(c) Write the matrices σ_i both in the form of Eq.(1) and their spectral decomposition in the Dirac notation.

Now, let us suppose that $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T \in \mathbb{R}^3$ is a vector of norm $|\vec{\alpha}| = 1$. Then $\vec{\alpha} \cdot \vec{\sigma} = \sum_{i=1}^{3} \alpha_i \sigma_i$ is a complex 2×2 matrix.

(d) Calculate the eigenvalues and eigenvectors of this matrix.

2. Probabilities (presence task)
We consider the random variables X, Y with possible outcomes $\Omega_X = \{0, 20, 25\}$, $\Omega_Y = \{0, 5, 10\}$ and $\Omega_{XY} = \Omega_X \times \Omega_Y$. The following probabilities are known:

$$p_x(0) = \frac{1}{2}, p_x(20) = p_x(25) = \frac{1}{4},$$

$$p_y(0) = \frac{1}{2}, p_y(5) = p_y(10) = \frac{1}{4}$$

and

$$p_{xy}(0,5) = p_{xy}(0,10) = p_{xy}(20,0) = p_{xy}(25,0) = \frac{1}{4}.$$

(a) Calculate the expectation values $E(X)$, $E(Y)$ and $E(XY)$.

Now suppose that $\Omega_X = \{50, 100\}$, $\Omega_Y = \{2, 10\}$ and $\Omega_{XY} = \Omega_X \times \Omega_Y$ with

$$p_x(50) = p_x(100) = \frac{1}{2},$$

$$p_y(2) = p_y(10) = \frac{1}{2}$$

and

$$p_{xy}(100,2) = p_{xy}(50,10) = p_{xy}(100,10) = p_{xy}(50,2) = \frac{1}{4}.$$

(b) Calculate again the expectation values $E(X)$, $E(Y)$ and $E(XY)$. What do you observe?

3. Polarization identity and positive semi-definite operators $(10 = 5 + 5)$

(a) We consider an operator A (not necessarily hermitian) in a complex Hilbert space $\mathcal{H}$. Prove that A is uniquely determined if the matrix element $\langle\psi|A|\psi\rangle$ is known for all states $|\psi\rangle \in \mathcal{H}$. I.e. all matrix elements $\langle\alpha|A|\beta\rangle$ can be expressed in terms of matrix elements of the form $\langle\psi|A|\psi\rangle$.

Hint: Consider the expectation value of A for the states $|\psi_1\rangle = |\alpha\rangle + |\beta\rangle$, $|\psi_2\rangle = |\alpha\rangle - |\beta\rangle$,$|\psi_3\rangle = |\alpha\rangle + i\,|\beta\rangle$ and $|\psi_4\rangle = |\alpha\rangle - i\,|\beta\rangle$. The states do not have to be normalized.

(b) Now suppose that $A \geq 0$ is a positive semi-definite operator. Prove that A is hermitian.

4. Antisymmetric Bell state $(10 = 3 + 3 + 4)$

From the lecture you already know the entangled state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|0\rangle_{\mathcal{A}} \otimes |1\rangle_{\mathcal{B}} - |1\rangle_{\mathcal{A}}\,|0\rangle_{\mathcal{B}})$, which is a so called Bell state. In this exercise we will check some useful properties of this state.

(a) Prove that for the two observables $A = \vec{\alpha} \cdot \vec{\sigma}$ and $B = \vec{\beta} \cdot \vec{\sigma}$ the expectation value takes the form $\langle\psi^-|A \otimes B|\psi^-\rangle = -(\vec{\alpha} \cdot \vec{\beta})$.

(b) Assume that the observable $A = \sigma_1$ is measured on the system $\mathcal{A}$ and the measurement results in the outcome $+1$. Determine the state of system $\mathcal{B}$ after the measurement. What happens if the measurement results in -1?

(c) The state $|\psi^-\rangle$ has the interesting property to be invariant (up to a phase) under arbitrary local unitary transformations, i.e. $U \otimes U\,|\psi^-\rangle = e^{i\phi}\,|\psi^-\rangle$. Prove this relation.