Theory of Quantum Matter

Lecturer: Prof. Otfried Gühne (Mon 14:00, Fri 10:00, Room D120)
Exercises: Chau Nguyen (Fri 12:30, Room D201) **Sheet 9**

Hand in: Mon 17.12.2018 *(questions marked as $*$ are optional)*
Discussion date: Fri 21.12.2018

14. The Fermi energy of impure graphene

(10pts) Recall that near the Dirac points (or Hamilton points), the energy band of graphene is linear, $E_\tau(\vec{k}) = \pm\hbar v_F k$, where v_F is known as the Fermi velocity of graphene, $v_F \approx 10^6$m/s and $\tau \in \{K, K'\}$ indicates the two inequivalent Dirac points. For pure graphene, exactly the lower band is filled; the Fermi energy is exactly zero. However when introducing impurity into the crystal (called *dopping*), additional electrons can be added to the system and the Fermi energy can increase above zero. Compute the lifted Fermi energy E_F as a function of the additional electron density n.

15. A fictitious 2D electron gas

(10pts bonus) Consider a fictitious 2D electron gas with the dispersal relation $\epsilon_{\vec{k}} = \hbar v_F k$. You can think of it as graphene, but here we ignore the valence band, different Dirac points and spins. The gas consists of N particles confined in an area S with periodic boundary condition. Compute the energy of the non-interacting gas in the ground state.

16. Mini Hubbard model

Consider the Hubbard model with Hamiltonian

$$H = -t \sum_{\langle i,j \rangle,\sigma} (a^\dagger_{i\sigma} a_{j\sigma} + \text{h.c.}) + U \sum_i n_{i\uparrow} n_{i\downarrow}. \tag{1}$$

(a) (10pts) Show that the total spin $S^z = \frac{1}{2}\sum_{i,\alpha,\beta} \sigma^z_{\alpha\beta} a^\dagger_{i\alpha} a_{i\beta}$ commutes with the Hamiltonian.
Hint: Write S^z in terms of $\sum_i n_{i\uparrow}$ and $\sum_i n_{i\downarrow}$ (interpret each term!). The potential obviously commutes with S^z. The kinetic terms (interpret each term!) move the particles without flipping their spins, thus should also commute with S^z; translate this intuition into a proof.

Now we suppose that the system has only two sites and is filled by two electrons.

(b) (5pts) Enumerate states in the occupation number basis and group them according to the eigenvalues of S^z. Indicate the ground state space at $t = 0$.

(c) (15pts) Write down the matrix elements of the full Hamiltonian in the occupation number basis. Note that S^z commutes with H, thus H is block-diagonal in the eigenspaces of S^z. Diagonalise the full Hamiltonian, possibly with Mathematica or other softwares. Indicate the ground state of the system and relate it to the description in the lectures. Is the ground state even or odd under permuting the two sites?