Theory of Quantum Matter

Lecturer: Prof. Otfried Gühne (Mon 14:00, Fri 10:00, Room D120)
Exercises: Chau Nguyen (Fri 12:30, Room XXXX) **Sheet 7**

Hand in: Mon 04.12.2018 *(questions marked as $*$ are optional)*
Discussion date: Fri 07.12.2018

11. Elementary manipulations of the creators and annihilators

(a) (5pts) Prove the following identities (which resemble the Leibniz derivation rule)

$$[A, BC] = B[A, C] + [A, B]C \tag{1}$$

$$[AB, C] = A[B, C] + [A, C]B \tag{2}$$

(b) (5pts) A monomial of operators $c^\dagger$ and c is said to be in *normal order* if all creators are to the *left* of annihilators. Bring the following monomials into sums of normal ordered monomials in the cases $c^\dagger$ and c are bosonic or fermionic operators: (i) $c_i c_j^\dagger$; (ii) $c_i c_j^\dagger c_k c_l^\dagger$.

(c) (5pts) For bosonic operators $b^\dagger$, b, show that $[b^\dagger, b^n] = -nb^{n-1}$ and $[b^\dagger, e^{tb}] = -e^{tb}$.

(d) (5pts) Let $c_1^\dagger$, c_1 and $c_2^\dagger$, c_2 be either bosonic or fermionic generators (your choice). Define $S^i = \sum_{\alpha,\beta=1}^{2} \sigma^i_{\alpha\beta} c^\dagger_\alpha c_\beta$ where σ^i are the Pauli matrices. Using the commutation relations for c, $c^\dagger$, show that S^i obeys the commutation relations of angular momentum operators, i.e., $[S^i, S^j] = i\sum_{k=1}^{3} \epsilon_{ijk} S^k$, where ϵ_{ijk} is the Levi-Civita symbol.

12. Second quantisation treatment of the free electron gas

Recall that the Hamiltonian for the free electron gas with periodic boundary box of volumn V in second quantisation is given by

$$H = \sum_{\vec{k}\alpha} \epsilon_k n_{\vec{k}\alpha}, \tag{3}$$

where $n_{\vec{k}\alpha} = a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha}$, $\epsilon_k = \frac{(\hbar k)^2}{2m}$ and the summation is running over all momenta quantised due to the periodic boundary condition and the spin indices α. Recall that the Hamiltonian commutes with the number operator $N = \sum_{\vec{k}\alpha} n_{\vec{k}\alpha}$, thus can be diagonalised at fixed number of particle N. In fact, the ground state of the Hamilotian is given by

$$|\Omega\rangle = \prod_{k \le k_F, \alpha} a^\dagger_{\vec{k}\alpha} |0\rangle , \tag{4}$$

where $|0\rangle$ is the vacuum state and the parameter k_F is called the Fermi momentum.

(a) (10pts) From the definition of the number operator given above, compute the number of particles in the ground state $\langle\Omega| N |\Omega\rangle$ in terms of k_F. Reverse this relation to find k_F as a function of the number of particles N (here N is a number).

Remark: Although you can compute things in first quantisation supplemented with the Pauli's exclusion principle, you are required to do the computation coherently in second quantisation. You can do this by figuring out how $n_{\vec{k}\alpha}$ acts on $|\Omega\rangle$.

(b) (10pts) From the Hamiltonian given above, compute the ground state energy $\langle\Omega| H |\Omega\rangle$. Write the ground state energy in terms of the particle number instead of k_F.

(c) ($*$) Compute the pressure exerted by the gas on the wall of the box.