Quantum Theory of Light (WS17/18)

Exercises 1

(For exercise class on Thu, November 24st 2017.)

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1. **Casimir effect.**
   Calculate the force on two parallel mirrors with dimensions $L \times L$ and distance d due to the vacuum energy of the electromagnetic field.

   Instructions.

   (a) Find a formal expression for the vacuum energy in a box of dimensions $L \times L \times d$ of the form

   $$E_{\text{vac}} \propto \sum_{\vec{k} \geq 0} |\vec{k}|,$$

   Remember to take polarization into account.

   (b) Implement the assumption that the modes are continuous in $k_x \equiv j_x \pi / L$ by replacing

   $$\sum_{j_x} \quad \longrightarrow \quad \int \mathrm{d}j_x.$$

   Proceed analogously with k_y. Regularize the integrand by $\exp(-\epsilon|\vec{k}|)$ and integrate. (Hint: Use polar coordinates.)

   (c) Compare the situation where the mirrors actually restrict the admissible modes with the situation where this is not the case and the modes in k_z are continuous. The difference is of the form

   $$U \propto -\frac{1}{2} f(0) + \sum_{j_z=0}^{\infty} f(j_z) - \int_0^{\infty} f(j_z)\,\mathrm{d}j_z,$$

   where $f(j_z)$ is the result of the regularized integral in the previous step. What is the reason for the first term?

   (d) Use the Euler–Maclaurin formula,

   $$\sum_{k=a+1}^{b} f(k) - \int_a^b f(k)\,\mathrm{d}k = -\frac{1}{2}(f(b) - f(a)) + \frac{1}{12}(f'(b) - f'(a)) - \frac{1}{720}(f'''(b) - f'''(a)) + \cdots,$$

   to show that $U = -\hbar\pi^2 c L^2/(720 d^3)$ when $\epsilon \to 0$. Hence the Casimir force on the plates is given by

   $$F = -L^2 \frac{\hbar \pi^2 c}{240 d^4}.$$

Please turn over.

2. **Coherent states.**

   (a) Compute the commutator $[\hat{a}, D(\alpha)]$ using the definition of the displacement operator $D(\alpha)$.

   (b) Characterize the full set of right-eigenstates of $\hat{a}$, i.e., the solutions of $\hat{a}\,|\psi\rangle = \lambda\,|\psi\rangle$ with $\langle\psi|\psi\rangle = 1$ and $\lambda \in \mathbb{C}$. (Hint: The Poisson distribution is normalized.)

   (c) In contrast, characterize the full set of right-eigenstates of $\hat{a}^\dagger$.

   (d) Make sense of and prove the following relations

$$a^\dagger|\alpha\rangle\langle\alpha| = \left(\alpha + \frac{\partial}{\partial\alpha}\right)|\alpha\rangle\langle\alpha| \quad \text{and} \quad (-1)^{\hat{n}}\,|\alpha\rangle = -\,|\alpha\rangle\,.$$

3. **Phase states.**
   It is sometimes useful to define phase states as

$$|\varphi_m\rangle = (r+1)^{-\frac{1}{2}}\sum_{n=0}^{r} \mathrm{e}^{in\varphi_m}\,|n\rangle\,,$$

   where $\varphi_m = 2\pi m/(r+1)$ and r is some large natural number.

   (a) Show that the phase states form an orthonormal basis of the $(r+1)$-dimensional space spanned by the number states $\{|0\rangle\,, |1\rangle\,, \dots, |r\rangle\}$.

   (b) The Fourier transform of the number-representation of a state, $|\psi\rangle = \sum \psi(n)\,|n\rangle$, induces the phase representation

$$P(\phi) = \frac{1}{2\pi}\left|\sum_{n=0}^{\infty} \psi(n)\mathrm{e}^{-in\phi}\right|^2\,.$$

   Verify, that $P(\phi)\mathrm{d}\phi$ is the limiting distribution of $P(\varphi_m) = |\langle\varphi_m|\psi\rangle|^2$ for $r \to \infty$.

   (c) Use the phase representation to compute the phase variance, $(\Delta\phi)^2$, of a number state $|\psi\rangle = |n\rangle$. Is there an uncertainty relation between $\hat{n}$ and the phase ϕ?

   (d) Use numerical methods to plot $P(\phi)$ for the coherent states with $\alpha = 1$ and $\alpha = 3$. How does the plot change, if a complex phase is added to α?

4. **Squeezed states.**
   Calculate the expectation value of the combined rotated quadrature operator,

$$\mathcal{Y} \equiv Y_1 + iY_2 = \mathrm{e}^{-i\vartheta/2}(X_1 + iX_1),$$

   for the squeezed state $|\alpha, \zeta\rangle = D(\alpha)S(r\mathrm{e}^{i\vartheta})\,|0\rangle$.

5. **The Glauber–Sudarshan P-function.**

   (a) Use the definition of the P-function,

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha,$$

   to show that

$$\langle(a^\dagger)^m a^n\rangle = \int P(\alpha)(\alpha^*)^m\alpha^n\,\mathrm{d}^2\alpha.$$

   (b) Calculate the P-representation of single-mode thermal light. (Hint: Express ρ in terms of $\langle\hat{n}\rangle$.)

   (c) Use the previous two results to calculate the corresponding photon number variance, $(\Delta\hat{n})^2$.