Mathematical Methods in Quantum Theory WS 2020/2021

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Sheet 3 — Hand-Out: Wed, 02.12.2020 — Due: Wed, 09.12.2020

Exercise 9: Duals of quotient spaces

Let X be a normed space and $V \leq X$ a subspace of X. Prove the following statements:

(a) $(X/V)^* \simeq V^\perp$

(b) $V^* \simeq X^*/V^\perp$

Exercise 10: Positive operators and product spaces

(a) Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. Show that the following statements are equivalent:

   (1) $A = A^*$ and $\sigma(A) \subset \mathbb{R}_+$

   (2) $\langle \varphi | A | \varphi \rangle \geq 0 \ \forall \varphi \in \mathcal{H}$

   (3) $\exists X \in \mathcal{B}(\mathcal{H})$ such that $A = X^* X$

(b) Let $(X, ||\cdot||_X)$ be a Banach space and let $X_1, X_2 \leq X$ be closed subspaces such that $X = X_1 \oplus X_2$. Prove that there exists a continuous linear isomorphism between X and $X_1 \times X_2$ with a continuous inverse. Note that the product space $X_1 \times X_2$ is equipped with the norm $||(x_1, x_2)|| = ||x_1||_X + ||x_2||_X$

Hint: Here the best way to prove that there is such a map $\varphi : X_1 \times X_2 \to X$ is by explicit construction.

Exercise 11: A basis for $\mathbf{L}^2([0,1))$

Consider the set $\mathcal{J}_0 := \{[0,1)\}$, which consists of the half-open interval $[0,1)$. Further we recursively define for any $k \in \mathbb{N}$ the sets $\mathcal{J}_k$ as the collection of those intervals obtained from splitting each interval in $\mathcal{J}_{k-1}$ in half, i.e.,

$$\mathcal{J}_2 := \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\} \ , \ \mathcal{J}_3 := \{[0, \frac{1}{4}), [\frac{1}{4}, \frac{1}{2}), [\frac{1}{2}, \frac{3}{4}), [\frac{3}{4}, 1)\} \tag{1}$$

and we set $\mathcal{J} := \cup_{k \in \mathcal{N}} \mathcal{J}_k$. For each interval $I \in \mathcal{J}$ we denote its left half by I_+ and its right half by I_-.

(a) Show that for $I, J \in \mathcal{J}$ with $I \neq J$ we have either $I \cap J = \emptyset$, $I \subset J$ or $J \subset I$. Further if $I \subset J$ show that we have either $I \subset J_+$ or $I \subset J_-$.

In addition we define for each $I \in \mathcal{J}$ the function

$$h_I := \frac{1}{\lambda^1(I)^{\frac{1}{2}}}(\chi_{I_+} - \chi_{I_-}) \tag{2}$$

where λ^1 is the 1-dimensional Lebesgue measure and χ_{I_+}, χ_{I_-} are the respective indicator functions of the intervals I_+, I_-. Finally set $h_0 = \chi_{[0,1)}$ and define $\mathcal{F} := \{h_I \,|\, I \in \mathcal{J}\} \cup \{h_0\}$.

(b) Let $h, g \in \mathcal{F}$. Show that

$$\langle h|g\rangle = \begin{cases} 1 & \text{if } h \neq g \\ 0 & \text{if } h = g \end{cases} \tag{3}$$

where $\langle \cdot|\cdot\rangle : \mathrm{L}^2([0,1)) \to \mathbb{C}$ is the standard inner product on $\mathrm{L}^2([0,1))$.

(c) Let $I \in \mathcal{J}$. Show $\chi_I \in \operatorname{span}(\mathcal{F})$.

(d) Let $0 \leq a < b \leq 1$. Show that $\chi_{[a,b)} \in \operatorname{span}(\mathcal{F})$.

(e) Conclude that the span of $\mathcal{F}$ is dense in $L^2([0,1))$. You can use the fact that the span of indicator functions $\chi_{[a,b)}$ where $0 \leq a < b \leq 1$ is dense in $L^2([a,b))$.