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Sheet 2 — Hand-Out: Fr, 20.11.2020 — Due: Wed, 02.12.2020

Exercise 5: Completeness of normed spaces

(a) Let $I := [a, b] \subset \mathbb{R}$ an interval with $a < b$ and consider the family of vector spaces

$$\mathsf{Poly}_n := \{f : I \to \mathbb{R} \,|\, f \text{ is polynomial with} \deg(f) \leq n\}$$

Show that $\mathcal{P} := \cup_{n\in\mathbb{N}} \mathsf{Poly}_n$ becomes by virtue of

$$||f||_\infty := \sup_{x\in I} |f(x)|$$

a normed space which is *not* complete.

(b) Let $(X, \|\cdot\|)$ be a normed space [1].

(i) If $(X, \|\cdot\|)$ is complete and $Y \subset X$ closed, then also $(Y, \|\cdot\|)$ is a complete normed space. In short: a closed subset of an complete normed space inherits the completeness from the ambient space.

(ii) If $Y \subset X$ and $(Y, \|\cdot\|)$ is complete, then Y is closed in X

Exercise 6: Completeness of function spaces

Let $S \subset \mathbb{R}^n$ and $(Y, ||\cdot||)$ a Banach space. For $0 < \alpha \leq 1$ and $f : S \to Y$ we define

$$H_\alpha(f, S) := \sup\Big\{\frac{|f(x) - f(y)|}{|x - y|^\alpha} \,|\, x, y \in S\,,\, x \neq y\Big\} \in [0, \infty]$$

If $\Omega \subset \mathbb{R}^n$ open and bounded we define for $m \geq 0$ the function spaces

$$C^{m,\alpha}(\overline{\Omega}, Y) := \{f \in C^m(\overline{\Omega}, Y) \,\big|\, H_\alpha(\partial^s f, \overline{\Omega}) < \infty \text{ for } |s| = m\}$$

Prove that $(C^{m,\alpha}(\overline{\Omega}, Y), ||\cdot||_{C^{m,\alpha}(\overline{\Omega})})$ is a Banach space when

$$||f||_{C^{m,\alpha}(\overline{\Omega})} := \sum_{|s|\leq m} ||\partial^s f||_{C^0(\overline{\Omega})} + \sum_{|s|=m} H_\alpha(\partial^s f, \overline{\Omega})$$

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[1] The claims remain true also if one replaces the normed space by an metric space

Exercise 7: POVM

As you have already learned in the lecture the qubit state space can be seen as a 3-dimensional unit ball and thus any state can be parametrized by two variables, i.e., $|\psi\rangle = |\psi(\theta, \varphi)\rangle$.

(a) Show that the set of effects $\mathcal{M} = \{E_0, E_1, E_2\}$ is a valid POVM if $\varphi_k = \frac{\pi}{2}$ for all $k = 0, 1, 2$ and

$$E_k = \frac{2}{3}|\psi_k(\theta_k, \varphi_k)\rangle\langle\psi_k(\theta_k, \varphi_k)| \text{ with } \theta_k = \frac{2\pi k}{3} \text{ for } k = 0, 1, 2$$

(b) Compute the probability distribution for the outcomes of the measurement $\mathcal{M}$ if the states are given by

$$\rho_1 = \frac{1}{2}\mathbb{I}_2 \ , \ \rho_2 = \frac{1}{3}|+\rangle\langle+| + \frac{2}{3}|1\rangle\langle1| \ , \ \rho_3 = |\psi(\frac{\pi}{2}, 0)\rangle\langle\psi(\frac{\pi}{2}, 0)|$$

Exercise 8: Geometric properties of Hilbert spaces

Let X be a complex vector space, $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{C}$ a positive semidefinite hermitian form and $||x|| := \sqrt{\langle x, x\rangle}$.

(a) For $x, y \in X$ and $\alpha \in \mathbb{C}$ prove the following statements:

(1) $||\alpha x|| = |\alpha|\, ||x||$

(2) $|\langle x, y\rangle| \leq ||x||\, ||y||$

(3) $||x + y|| \leq ||x|| + ||y||$

(4) $||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$

(b) For $x, y \in X$ prove the polarisation identity

$$\langle x, y\rangle = \sum_{k=0}^{3} i^k ||x + i^k y||^2$$

(c) Show that if $(X, ||\cdot||)$ is a normed space and $(a4)$ holds then there exists an inner product $\langle\cdot,\cdot\rangle$ on X such that for all $x \in X$ we have $||x||^2 = \langle x, x\rangle$.