Mathematical Methods in Quantum Theory WS 2020/2021

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Sheet 1 — Hand-Out: Fr, 13.11.2020 — Due: Fr, 20.11.2020

Exercise 1: Convex Geometry

Let X be a convex set. Prove the following statements:

(a) If $\xi \in X$ can be written as a convex combination of n elements that is

$$\xi = \sum_{i=1}^{n} \lambda_i \xi_i \quad \text{with} \quad \sum_{i=1}^{n} \lambda_i = 1 \ , \ \lambda_i > 0$$

then it can also be written as a convex combination of two elements, i.e., one can find $\lambda \in [0,1]$ and $\rho_1, \rho_2 \in X$ such that $\xi = \lambda\rho_1 + (1-\lambda)\rho_2$.

(b) Suppose that $\xi \in X$ is not extremal. Then one can find $\xi_1, \xi_2 \in X$ with $\xi_1 \neq \xi_2$ such that $\xi = \frac{1}{2}\xi_1 + \frac{1}{2}\xi_2$.

Exercise 2: Classical State Space

Consider the set of probability distributions on a finite set $\{1, ..., n\}$, i.e., the set

$$\mathcal{P}_n = \{\vec{p} \in \mathbb{R}^n_+ \,|\, p_j \geq 0 \ , \ \sum_{j=1}^{n} p_j = 1\}$$

(a) Show that $\mathcal{P}_n$ is a convex set and determine its extremal elements

(b) Is the decomposition of $\xi \in \mathcal{P}_n$ into extremal elements unique?

(c) Compute the set of effects $\mathcal{E}_n$.

Exercise 3: Semidefinite linear programs

In quantum information one often encounters optimization problems of the following form: Let $F_0, ..., F_n \in \mathbb{R}^{n\times n}$ be symmetric matrices and $c, d \in \mathbb{R}^n$. A semidefinite program (SDP) is an optimization problem where the objective is a linear function of the variables, and the constraints consist of linear matrix inequality (LMI) constraints, and linear equality constraints, i.e.,

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & F_0 + F_1 x_1 + ... + F_n x_n \geq 0 \\ & Ax = b \end{aligned}$$

(a) Show that the set of $x \in \mathbb{R}^n$ satisfying the constraints is convex

(b) Represent the linear equality constraints $a_i^\top x = b_i$ for $i = 1, ..., k$ with variable x as an LMI

Exercise 4: Spin-factors

As you have already learned in the lecture, the state space of a quantum system with two degrees of freedom can be described via the Bloch ball, that is, the unit ball in three dimensions. If one now changes the number of degrees of freedom from two to three (the so called qutrit), one would maybe expect that the state space is still a unit ball but in higher dimensions. Unfortunately, it turns out that this is not true. In fact, the qutrit state space has a much richer geometrical structure compared to the qubit ones. To understand the origin of this structure, one typically considers so called toy theories, i.e., abstract physical theories that are designed for the purpose of structural analysis. One popular family of examples are the spin factors [1] where the state space is defined via

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \, : \, ||x|| \leq 1\} \text{ with } ||x|| = \sqrt{\sum_{i=1}^{n} |x_i|}$$

and $\mathbb{S}^{n-1}$ is the n-sphere. Prove the following statements:

(a) $x \in \mathbb{S}^{n-1}$ is an extreme point of $\mathbb{S}^{n-1}$ if and only if $||x|| = 1$

(b) Any (nonextremal) element $x \in \mathbb{S}^{n-1}$ can be written as a convex combination of two extremal points.

[1] This spin factors are not only interesting because they are very similar to the state space of the qubit. They also arise as a representation of a special class of algebraic objects, the so called Jordan-algebras. For further reading we refer to *Symmetry, Self-Duality and the Jordan-structure of Quantum Theory, arXiv:1110.6607*