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Sheet 4 — Hand-Out: Wed, 09.12.2020 — Due: Wed, 23.12.2020

Exercise 12: Tensor products

Decide if the following states are separable or entangled and write them as vectors in the standard basis:

(a) $|\psi\rangle = \frac{1}{\sqrt{2}}(\alpha|00\rangle + i\beta|11\rangle + i\alpha|01\rangle + \beta|10\rangle)$ with $\alpha, \beta \in \mathbb{C}$ and $|a|^2 + |\beta|^2 = 1$

(b) $|\psi\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)$

(c) $|\psi\rangle = \frac{1}{2}(|000\rangle + |010\rangle - |001\rangle - |011\rangle)$

Write $A \otimes B$ and $B \otimes A$ in the standard basis

(d) $A = \alpha\sigma_x + \beta\sigma_y$ and $B = \gamma\sigma_z + \mathbb{I}$

(e)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;, \; B = \begin{pmatrix} a & \alpha & \beta \\ \alpha^* & b & \gamma \\ \beta^* & \gamma^* & c \end{pmatrix}$$

(f) $A = a|+\rangle\langle+| + b|-\rangle\langle-|$ and $B = \sigma_y \otimes \sigma_y$

(g) Show that $(A \otimes A)|\Psi^-\rangle = \det(A)|\Psi^-\rangle$, where $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and $A \in \mathrm{M}_2(\mathbb{C})$. What does this result imply for the case when $A \in \mathrm{U}_2(\mathbb{C})$ is unitary and what is the physical interpretation of this property in that case? Does there exist another two qubit pure state with that property?

Exercise 13: Partial trace

Let $T \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a trace class operator and $|\phi\rangle \in \mathcal{H}_A$. Prove the following statements

(a) For every orthonormal basis $\{|\psi_k\rangle\}_k$ of $\mathcal{H}_B$ one has

$$\langle\phi|\mathrm{tr}_B(T)|\phi\rangle = \sum_k \langle\phi \otimes \psi_k|T|\phi \otimes \psi_k\rangle$$

(b) $\mathrm{tr}(T) = \mathrm{tr}(\mathrm{tr}_A(T)) = \mathrm{tr}(\mathrm{tr}_B(T))$

(c) $T \geq 0$ implies $\mathrm{tr}_A(T) \geq 0$ and $\mathrm{tr}_B(T) \geq 0$

(d) Compute the partial trace (both tr_A and tr_B) for the following states

(i) $\rho = \frac{1}{4}(|00\rangle\langle 00| + |10\rangle\langle 10| + |01\rangle\langle 01| + |11\rangle\langle 11|)$

(ii) $\rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$

(iii) $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

Exercise 14:

Let $\mathcal{H}$ be a Hilbert space with $\dim(\mathcal{H}) = n < \infty$ and $A, B \in \mathcal{B}(\mathcal{H})$. Show the following:

(a) $(A \otimes B)^* = A^* \otimes B^*$, $(A \otimes B)^\top = A^\top \otimes B^\top$ and $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$

(b) If $A, B \in \mathrm{U}(\mathcal{H})$, then $A \otimes B \in U(\mathcal{H} \otimes \mathcal{H})$

(c) If A, B hermitian, then $A \otimes B$ hermitian

(d) If $A, B \geq 0$, then $A \otimes B \geq 0$

(e) If A, B are projectors, then $A \otimes B$ is a projector

Let A_1, A_2, A_3 be complex Hilbert spaces with $\dim(A_i) = n_i < \infty$ for $i = 1, 2, 3$. Prove the following

(f) $A_1 \otimes A_2 \simeq A_2 \otimes A_1$

(g) $\mathbb{C} \otimes A_1 \simeq A_1$

(h) $A_1 \otimes (A_2 \otimes A_3) \simeq (A_1 \otimes A_2) \otimes A_3$

(i) $(A_1 \oplus A_2) \otimes A_3 \simeq (A_1 \otimes A_3) \oplus (A_2 \otimes A_3)$