Quantum theory of light

Lecturer: Matthias Kleinmann (Tue 14:15, Room B030)
Exercises: Chau Nguyen (Mon 16:15, Room D120) **Sheet 9**

Hand in: Tue 07.01.2020 *(questions marked as ∗ are optional)*
Discussion date: Mon 13.01.2020

21. Some more about the Rabi model

[5pts] Recall that in the Rabi model the two levels $|i\rangle$ and $|f\rangle$ of the atom couple to the electromagnetic wave $\vec{E}_0 \cos \omega t$ by the dipole approximation $H_I = -\vec{r} \cdot \vec{E}_0 \cos \omega t$. One assumes that $\langle i| \vec{r} \cdot \vec{E}_0 |i\rangle = \langle f| \vec{r} \cdot \vec{E}_0 |f\rangle = 0$ and $\langle f| \vec{r} \cdot \vec{E}_0 |i\rangle = \beta$. Using the solution to the Rabi model in the rotating wave approximation in the lecture, compute and plot the dipole coupling over time $\left\langle \vec{r}(t) \cdot \vec{E}_0 \right\rangle$.

22. Dark state

Consider an atom which has three neighbouring states, $|1\rangle$, $|2\rangle$ and $|3\rangle$ with energies $\hbar\omega_1 < \hbar\omega_2 < \hbar\omega_3$. The transitions $|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |3\rangle$ are allowed by dipole coupling, but $|1\rangle \leftrightarrow |2\rangle$ is forbidden. Under the interaction with two light fields (the pumping field and the probing field), the system Hamiltonian in the rotating wave approximation is

$$H = \sum_{i=1}^{3} \hbar\omega_i |i\rangle \langle i| - \frac{\hbar}{2} \left[\left(\Omega_p e^{-i\omega_p t} |1\rangle \langle 3| + \Omega_c e^{-i\omega_c t} |2\rangle \langle 3| \right) + \text{h.c.} \right]. \tag{1}$$

where Ω_p and Ω_c are the Rabi frequencies tunned at exact resonances with $\omega_p = \omega_3 - \omega_1$ and $\omega_c = \omega_3 - \omega_2$.

(a) [5pts] Expand the state of the system in the interaction picture as $|\psi(t)\rangle_I = \sum_{i=1}^{3} c_i(t) |i\rangle$. Derive the Schrödinger equation for $c_i(t)$ explicitly.

(b) [5pts] Find a fixed point solution (solution that do not depend on time) for the system. Comment on the physical meaning of the solution.

(c) [∗, 5pts] Find the general solution for $c_i(t)$ subjecting to general initial condition.

23. Jaynes-Cummings model

[10pts] Consider the Jaynes-Cummings Hamiltonian with the interaction term

$$H_I = \hbar\beta a^\dagger a(\sigma_+ + \sigma_-). \tag{2}$$

Compute the atomic inversion $W(t)$ for the system with the initial state $|g, \alpha\rangle$, i.e., the atom is in the ground state and the field is in a coherent state.