Quantum theory of light

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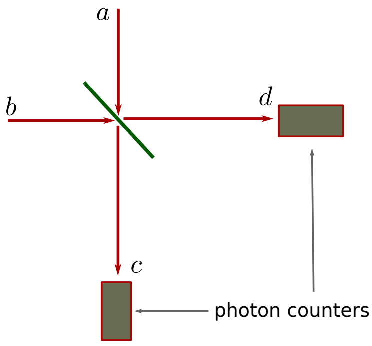
Exercises: Chau Nguyen (Mon 16:15, Room D120) **Sheet 6**

Hand in: Tue 03.12.2019 *(questions marked as $*$ are optional)*
Discussion date: Mon 09.12.2019

14. Demonstrating squeezing by measuring the quadratures

In this problem we describe how the squeezed states are detected by means of the *balanced homodyne* setup. In this setup, the squeezed mode a is mixed with a mode b by a beam-splitter (see figure beside). The beam-splitter implements the transformation

$$c = \frac{1}{\sqrt{2}}(a+ib), d = \frac{1}{\sqrt{2}}(b+ia). \tag{1}$$



Suppose that all the modes are *of the same frequency* ω (they are all derived from the same laser in practice.) The modes c and d are the subjected to photon counters which measure the photon numbers n_c and n_d and denote $n_{cd} := n_c - n_d$.

(a) [5pts] Show that $n_{cd} = i(a^\dagger b - ab^\dagger)$.

(b) [5pts] Suppose mode b is in a large coherent state $|\beta\rangle$ with $\beta = |\beta|e^{\theta}$, show that n_{cd} implements the measurement of a quadrature of mode a. The fluctuation thus allows one to determine the squeezed quadratures of the state.

15. Two-mode squeezing

In similarity to the one-mode squeezing operator, one can introduce the two-mode squeezing operator

$$S_2(\xi) = \exp(\xi^* ab - \xi a^\dagger b^\dagger) \tag{2}$$

with $\xi = re^{i\theta}$, where a and b are two different modes. The two-mode squeezed vacuum is given by $|\xi\rangle_2 = S_2(\xi)\,|0,0\rangle$.

(a) [5pts] Show that

$$S_2^\dagger(\xi) a S_2(\xi) = a\cosh r - e^{i\theta} b^\dagger \sinh r, \tag{3}$$

$$S_2^\dagger(\xi) b S_2(\xi) = b\cosh r - e^{i\theta} a^\dagger \sinh r. \tag{4}$$

(b) [10pts] Define the quadratures involving two modes $X_1 = 1/\sqrt{8}(a + a^\dagger + b + b^\dagger)$, $X_2 = 1/(\sqrt{8}i)(a - a^\dagger + b - b^\dagger)$. Show that $[X_1, X_2] = i/2$. And show that for the two-mode squeezed vacuum $|\xi\rangle_2$ with $\theta = 0$, $\langle X_1 \rangle = \langle X_2 \rangle = 0$ and $\langle X_1^2 \rangle = e^{-2r}/4$, $\langle X_2^2 \rangle = e^{2r}/4$.

(c) [$*$, 10pts] Show that the two-mode squeezed vacuum can be expanded in the number states as

$$|\xi\rangle_2 = \frac{1}{\cosh r}\sum_{n=0}^{+\infty}(-1)^n e^{in\theta}(\tanh r)^n\,|n,n\rangle\,. \tag{5}$$

Then show that the photon statistics of each mode is that of the thermal state with effective temperature $T_{\text{eff}} = \hbar\omega_i/(2k_B \ln\cosh r)$, where ω_i are the frequencies of the modes.

(c) [5pts] In the lecture, you learn how to produce a (one-mode) squeezed state by means of the *degenerate parametric down-conversion.* Following the same steps, describe how the two-mode squeezed vacuum can be generated by the *non-degenerate parametric down-conversion* with the Hamiltonian

$$H = \hbar\omega_a a^\dagger a + \hbar\omega_b b^\dagger b + \hbar\omega_p c^\dagger c + i\chi^{(2)}(abc^\dagger - a^\dagger b^\dagger c), \tag{6}$$

where c denotes the pumping mode, and $\chi^{(2)}$ is the coupling.