Quantum theory of light

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Exercises: Chau Nguyen (Mon 16:15, Room D120) **Sheet 5**

Hand in: Tue 19.11.2019 *(questions marked as $*$ are optional)*
Discussion date: Mon 25.11.2019

Recall that the characteristic functions of a state ρ are defined by

$$C_{\mathrm{W}}(\lambda) = \mathrm{Tr}(\rho e^{\lambda a^\dagger - \lambda^* a}) \text{ (Wigner)}, \tag{1}$$
$$C_{\mathrm{N}}(\lambda) = \mathrm{Tr}(\rho e^{\lambda a^\dagger} e^{-\lambda^* a}) \text{ (normally ordered)} \tag{2}$$
$$C_{\mathrm{A}}(\lambda) = \mathrm{Tr}(\rho e^{-\lambda^* a} e^{\lambda a^\dagger}) \text{ (antinormally ordered)} \tag{3}$$

Then the three (quasi)-probability distributions are given by

$$Q(\alpha) = \frac{1}{\pi^2}\int \mathrm{d}\lambda e^{\lambda^*\alpha - \lambda\alpha^*} C_{\mathrm{A}}(\lambda) \tag{4}$$
$$P(\alpha) = \frac{1}{\pi^2}\int \mathrm{d}\lambda e^{\lambda^*\alpha - \lambda\alpha^*} C_{\mathrm{N}}(\lambda) \tag{5}$$
$$W(\alpha) = \frac{1}{\pi^2}\int \mathrm{d}\lambda e^{\lambda^*\alpha - \lambda\alpha^*} C_{\mathrm{W}}(\lambda) \tag{6}$$

From the above definitions, one can show that

$$\rho = \int \mathrm{d}\alpha P(\alpha) \left|\alpha\right\rangle \left\langle\alpha\right|, \; Q(\alpha) = \left\langle\alpha\right| \rho \left|\alpha\right\rangle / \pi, \tag{7}$$

and

$$W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \mathrm{d}x \left\langle q + \frac{x}{2} |\rho| q - \frac{x}{2} \right\rangle e^{ipx/\hbar}, \tag{8}$$

where q and p are the position and momentum operators, and $|q \pm x\rangle$ are the eigenstates of the position operator.

12. Examples of the phase space representations

(a) [5pts] Compute the Q-function and P-function for the single mode thermal state $\rho = [1 - e^{-\hbar\omega/kT}]e^{-\hbar\omega a^\dagger a/kT}$.

(b) [10pts] The squeezed state is given by $|\alpha, \xi\rangle = D(\alpha)S(\xi)|0\rangle$, where $S(\xi) = \exp\left[\frac{1}{2}(\xi^* a^2 - \xi a^{\dagger 2})\right]$ (squeeze operator) and $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ (displacement operator). Compute the Wigner function for the squeezed state $|\alpha, \xi\rangle$ for the case $\xi = r$ is purely real. Describe how it evolves over time.

13. The squeezed state

(a) [5pts] Compute the Wigner function for the statistical mixture of two coherent states $\rho = \frac{1}{2}(\left|\alpha\right\rangle\left\langle\alpha\right| + \left|-\alpha\right\rangle\left\langle-\alpha\right|)$. Describe how it evolves over time.

(b) [10pts] Compute the Wigner functions for the optical cat states $|\mathrm{Meow}_\pm\rangle = 1/N_\pm(|\alpha\rangle \pm |-\alpha\rangle)$. Describe how they evolve over time.

(c) [$*$,5pts] Plot the Wigner functions obtained in (a) and (b) by a suitable computer program of your choice. Sketch the outcomes on your sheet. What are the difference between the two.