Quantum theory of light

Lecturer: Matthias Kleinmann (Tue 14:15, Room B030)
Exercises: Chau Nguyen (Mon 16:15, Room D120) **Sheet 2**

Hand in: Tue 29.10.2019 *(questions marked as ∗ are optional)*
Discussion date: Mon 04.11.2019

3. Einstein's argument for the Planck distribution

In a paper in 1917 (thus before the quantum mechanical theory of electrons and photons), Einstein gave a derivation of the Planck distribution law for equilibrium radiation. It is actually a nice one! Here is a simplified version of his argument. Consider a two-level system (without any other degree of freedom) with energies E_1 and E_0 in equilibrium with the radiation described by energy density per frequency $\rho(\omega)$. If the atom is in the ground state E_0, it can absorb a photon $\hbar\omega = E_1 - E_0$ to make the transition to state E_1. The rate of this transition is $\rho(\omega)B_{0\to1}$, with some factor $B_{0\to1}$. If the atom is in the excited state E_1, there are two possibilities. It can spontaneously decay with rate $A_{1\to0}$ to state E_0, emitting a photon $\hbar\omega = E_1 - E_0$. Here is the great insight from Einstein: the present radiation can also induce a transition from E_1 to E_0, say with rate $\rho(\omega)B_{1\to0}$.

(a) [5pts] The atom in equilibrium follows the Boltzman distribution, $p(E_i) \propto e^{-E_i/kT}$, where T is the temperature and k is the Boltzman constant. Show that the balance of the transition probabilities leads to

$$\rho(\omega)B_{0\to1} = [\rho(\omega)B_{1\to0} + A_{1\to0}]e^{-\frac{\hbar\omega}{kT}}. \tag{1}$$

Note that the Einstein coefficients $A_{1\to0}$, $B_{1\to0}$ and $B_{0\to1}$ can depend explicitly on the frequency ω, but not the temperature T.

(b) [5pts] Now (following Einstein), consider the limit $T \to \infty$. What do you expect $\rho(\omega)$ to be in this limit? Show that $B_{0\to1} = B_{1\to0}$.

Remark: This may come as a surprise! The radiation can fallicitate the transition between the two levels somewhat equally in both ways. Note that Einstein arrived at this before any formal development of QED.

(c) [5pts] It is also well-known at the time that for the short wavelength limit, the distribution follows the Wein's law

$$\rho(\omega) \propto \omega^3 e^{-\frac{\hbar\omega}{kT}}. \tag{2}$$

From this, derive Plack's law

$$\rho(\omega) \propto \frac{\omega^3}{e^{\frac{\hbar\omega}{kT}} - 1}. \tag{3}$$

4. Useful operator identities

(a) [5pts] Suppose $[X, Y]$ commutes with both X and Y, show that

$$e^{sX} Y e^{-sX} = Y + s[X, Y]. \tag{4}$$

(b) [5pts] Suppose $[X, Y]$ commutes with both X and Y, show that

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}. \tag{5}$$

Hint: Derive a differential equation for $G(s) = e^{sX}e^{sY}$, make proper use of question (a) and solve it.

(c) [5pts] For a and $a^\dagger$ being the bosonic anihilator and creator, show that

$$[a^\dagger, F(a)] = -F'(a) \text{ and } a^\dagger F(aa^\dagger) = F(a^\dagger a)a^\dagger, \tag{6}$$

for analytic (operator) function F.

5. Regularisation in the Casimir effect

(a) [*5pts] Familiarize yourself with the Euler–Maclaurin formula in your favourite analysis book. To which functions can it be applied?

(b) [*5pts] Assume that $g(n)$ is an appropriate function with $g(n) \approx 1$ for small n and $g(n) \approx 0$ for large n. Show that, independent of the details of $g(n)$,

$$\sum_{n=0}^{u} ng(n) - \int_0^u ng(n)\mathrm{d}n \approx -1/12$$

as $u \to \infty$. What properties does $g(n)$ need to have for this to work? Does a function like $g(x) = 1 - 1/[\exp((x_0 - x)/\epsilon) + 1]$ work?