Foundations of Quantum Mechanics

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Sheet 4 — SS 2019 — Due: Mon., 13.05.2019

Exercise 13: Von Neumann entropy (2+2+4)

The von Neumann entropy of a quantum state ϱ is defined by $S(\varrho) = -\operatorname{tr}[\varrho \log(\varrho)]$, where $\log(\cdot)$ denotes the matrix logarithm. If ϱ is given in terms if its eigenvectors $\varrho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$ the von Neumann entropy simplifies to $S(\varrho) = -\sum_i \lambda_i \log(\lambda_i)$. Show that:

a) The von Neumann entropy is non-negative and zero if and only if ϱ is pure.

b) In a d-dimensional Hilbert space the entropy is at most $\log(d)$ and the maximum is obtained if and only if the system is in the completely mixed state $\varrho = \mathbb{1}/d$.

c) Now consider a state $\varrho = \sum_i p_i \varrho_i$, where the ϱ_i have orthogonal support. Prove that
$$S\left(\sum_i p_i \varrho_i\right) = H(p_i) + \sum_i p_i S(\varrho_i), \tag{1}$$
where $H(p_i)$ is the Shannon entropy.

Exercise 14: Coexistence does not imply Joint Measurability (2+4+2)

Besides joint measurability there exists another notion of when POVMs can be measured together. Two POVMs $\mathbf{A}$ and $\mathbf{B}$ are called *coexistent*, if there exists a larger POVM $\mathbf{M}$ that contains the ranges of $\mathbf{A}$ and $\mathbf{B}$, i.e. $\operatorname{ran}(\mathbf{A}) \cup \operatorname{ran}(\mathbf{B}) \subseteq \operatorname{ran}(\mathbf{M})$. Consider the Hilbert space $\mathbb{C}^3$ with basis elements $\{|1\rangle, |2\rangle, |3\rangle\}$ and $|\psi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$. We define two POVMs
$$A_i = \frac{1}{2}(\mathbb{1} - |i\rangle\langle i|), \quad i = 1, 2, 3 \tag{2}$$
and
$$B_1 = \frac{1}{2}|\psi\rangle\langle\psi|, \quad B_2 = \mathbb{1} - B_1. \tag{3}$$

a) Show that the POVMs $\mathbf{A} = \{A_i\}_i$ and $\mathbf{B} = \{B_1, B_2\}$ are coexistent, so that the ranges of $\mathbf{A}$ and $\mathbf{B}$ are contained in the range of some larger POVM $\mathbf{M}$.

b) Now show that these POVMs are not jointly measurable. Hints: Try to prove this by contradiction. Assume the POVMs are jointly measurable. Then use the fact that B_1 is a rank-1 operator and derive conditions on the J_{i1}. Use this to express the A_i in terms of B_1 and compute the overlap $\langle i| A_i |i\rangle$. From this, find a contradiction.

c) Why is joint measurability and coexistance equivalent in the case of two dichotomic POVMs $\mathbf{A} = \{A_1, A_2\}$ and $\mathbf{B} = \{B_1, B_2\}$?