Foundations of Quantum Mechanics

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Sheet 3 — SS 2019 — Due: Mon., 06.05.2019

Exercise 10: Shannon entropy (2+2+2)

Show that the Shannon entropy fulfills the following properties:

a) For any probability distribution $\mathcal{P} = (p_1, \ldots, p_N)$ there exsists the upper bound $S(\mathcal{P}) \leq \log(N)$. The bound is saturated for $\mathcal{P}$ being the equal distribution;

b) The Shannon entropy is concave;

c) The Shannon entropy is additive, i.e. $S(\mathcal{P}) = S(\mathcal{Q}) + S(\mathcal{R})$, where $\mathcal{Q} = (q_1, \ldots, q_n)$ and $\mathcal{R} = (r_1, \ldots, r_m)$ are probability distributions and $\mathcal{P} = (p_{11}, p_{12}, \ldots, p_{1m}, p_{21}, p_{22} \ldots, p_{nm})$ for $p_{ij} = q_i r_j$.

Exercise 11: Uncertainty relations (6+3)

a) Derive the uncertainty relation

$$S(A) + S(B) \geq -2\log(c),$$

for the following examples.

i) $A = \sigma_x$, $B = \sigma_y$, $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$

ii) $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, $\varrho = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|2\rangle\langle 2|$, mit $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

iii) $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$, $B = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 3 \\ 0 & 0 & 3 & -1 \\ -1 & 3 & 0 & 0 \\ 3 & -1 & 0 & 0 \end{pmatrix}$, $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$

b) For each example, find the state such that the bound is optimal.

Please turn!

Exercise 12: Gaussian wave packet

Show that a Gaussian wave packet $\Psi(x) = Ne^{-\mu x^2+\nu x}$, where N is a normalisation constant, $\mu > 0$ and $\nu \in C$, is a state of minimal position-momentum uncertainty. Hints: $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ for $a > 0$ and $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$ for $a > 0$.