**Foundations of Quantum Mechanics**

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*Sheet 2* — SS 2019 — Due: Wed., 24.04.2019

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## Exercise 6: Hadamard's inequality

Show that for any positive definite matrix $M > 0$ the following inequality holds:

$$\prod_i m_{ii} \geq \det(M), \tag{1}$$

where $m_{ii} = (M)_{ii}$. Hint: Make use of the fact that for any positive definite $n \times n$ matrix $M$ and $D = \mathrm{diag}(\sqrt{m_{11}}, \ldots, \sqrt{m_{nn}}$ it holds that $D^{-1}MD^{-1} > 0$ and $\mathrm{tr}(D^{-1}MD^{-1}) = n$.

## Exercise 7: Unambiguous state discrimination

Suppose Alice is given a state from the ensemble $\mathcal{E} = \{\varrho_1, \varrho_2\}$ and she wants to guess which one it is without ever guessing wrong. This can be achieved by allowing her to not make a guess at all based on the result of the measurement she has performed.

a) Her measurement has at least two outcomes $\{1, 2\}$ (with POVM elements $E_1$ and $E_2$) corresponding to $\varrho_1$ and $\varrho_2$, respectively. Assume the two states are pure, i.e. $\varrho_j = |\phi_j\rangle\langle\phi_j|$, what is the general form of the $E_j$ such that $p(E_j|\varrho_k) = 0$ for $j \neq k$?

b) When do these elements alone form a valid POVM?

c) Assuming that both states are given to Alice with probability 1/2, show that choosing $\mathrm{tr}(E_1) = \mathrm{tr}(E_2)$ is optimal.

## Exercise 8: Density matrices

A set of states $\left\{\sqrt{p_i}, |\psi_i\rangle\right\}$, with $\{p_i\}$ a probability distribution, defines a density matrix $\varrho_\psi = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. A natural question to ask is the following: Given another (potentially different) set of states $\left\{\sqrt{q_j}, |\phi_j\rangle\right\}$ defining a density matrix $\varrho_\phi = \sum_j p_j |\phi_j\rangle\langle\phi_j|$, under which conditions are the resulting density matrices equal?

Show that $\varrho_\psi = \varrho_\phi$ if and only if $\sqrt{p_i}\,|\psi_i\rangle = \sum_j u_{ij}\sqrt{q_j}\,|\phi_j\rangle$, for some unitary matrix $u_{ij}$.

*Bitte wenden!*

**Exercise 9: Quantum Zeno effect**

At time $t = 0$ a qubit is in the state $|\psi\rangle = |0\rangle$. The time evolustion is given by the unitary

$$U(t) = \begin{pmatrix} \cos(\omega t) & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}. \tag{2}$$

(a) Calculate $\varrho(t)$. How does $\varrho(t)$ move on the Bloch sphere? What are the probabilities of obtaining $\pm 1$ when measuring $\sigma_z$ at time $t$?

(b) A sequence of $N$ $\sigma_z$ measurements is being performed, each after a short timestep ($\delta t \ll 1$). What is the probability to find the system in the initial state after $N$ measurements.

(c) At time $T$ a sequence of $N$ measurements of $\sigma_z$ has been performed. The timesteps were chosen to be $\delta t = T/N$. What happens in the limit $N \to \infty$? What is the probability to find the system in the initial state at time $T$? Hint: Use the formula

$$\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a. \tag{3}$$

(d) Consider the time evolution

$$\varrho(t) = e^{-\alpha t} |0\rangle\langle 0| + (1 - e^{-\alpha t}) |1\rangle\langle 1| . \tag{4}$$

Does the same effect occur in this case? Is this time evolution unitary?