Foundations of Quantum Mechanics

Prof. Dr. O. Gühne, T. Kraft

Sheet 10 — SS 2019 — Due: Mon., 24.06.2019

Exercise 26: Semidefinite programming (3+3+2)

In quanutm information many problems can be cast in terms of a so-called semidefinite program, i.e. a convex optimization problem over the cone of positive semidefinite matrices. Semidefinite programs always come in pairs, the primal problem and the dual problem. The generic primal problem is of the form

$$\begin{aligned} \max_X : \quad & \mathrm{Tr}[AX] \\ \text{such that} : \quad & \Phi(X) = B \; . \\ & X \geq 0 \end{aligned} \tag{1}$$

The corresponding dual problem reads

$$\begin{aligned} \min_Y : \quad & \mathrm{Tr}[BY] \\ \text{such that} : \quad & \Phi^*(Y) \geq A \\ & Y \text{ hermitian} \; . \end{aligned} \tag{2}$$

A and B are hermitian matrices and $\Phi(\cdot)$ is a hermicity preserving map. $\Phi^*(\cdot)$ denotes the adjoint map of Φ and it is defined though $\mathrm{Tr}[Y\Phi(X)] = \mathrm{Tr}\left[\Phi^*(Y)X\right]$.
Assume you are given an hermitian operator H, i.e. an observable, and you wish to compute the eigenstate which corresponds to the largest eigenvalue λ^{max} of H.

a) Cast the problem of finding this eigenstate as a (primal) semidefinite program.

b) Now calculate the corresponding dual program. What does it compute.

c) Do both optimizations result in the same value of the objective function?

Exercise 27: Hardy's test of quantum non-locality (3+5)

Consider an experiment, similar to the one of CHSH, where Alice measures observables X_1 or Y_1 and Bob measures observables X_2 or Y_2, all with outcomes ± 1. Suppose, an experimental setup is found in such a way that

$$\begin{aligned} P\left(+,+|X_1,X_2\right) &= 0, \\ P\left(+,-|Y_1,X_2\right) &= 0, \\ P\left(-,+|X_1,Y_2\right) &= 0. \end{aligned} \tag{3}$$

a) Assuming that nature is described by a local hidden variable theory, show that

$$P\left(+,+|Y_1,Y_2\right)=0 \tag{4}$$

must hold. *Hint:* Assume an event with $y_1=y_2=+1$ was detected for a simultaneous measurement of Y_1 and Y_2. Furthermore, use the locality assumption, namely that the value of x_2 cannot depend on whether Alice measured x_1 or y_1, and x_1 cannot depend on whether Bob measured x_2 or y_2.

b) Suppose Alice and Bob share the state $|\psi\rangle=b|01\rangle+c|10\rangle+d|11\rangle$. Alice can measure $X_1=|0\rangle\langle 0|-|1\rangle\langle 1|$ and $Y_1=\left|y_1^+\right\rangle\left\langle y_1^+\right|-\left|y_1^-\right\rangle\left\langle y_1^-\right|$, where

$$\left|y_1^+\right\rangle=\frac{d^*|0\rangle-b^*|1\rangle}{\sqrt{|b|^2+|d|^2}}, \tag{5}$$

and

$$\left|y_1^-\right\rangle=\frac{b|0\rangle+d|1\rangle}{\sqrt{|b|^2+|d|^2}}. \tag{6}$$

And Bob can measure $X_2=|0\rangle\langle 0|-|1\rangle\langle 1|$ and $Y_2=\left|y_2^+\right\rangle\left\langle y_2^+\right|-\left|y_2^-\right\rangle\left\langle y_2^-\right|$, where

$$\left|y_2^+\right\rangle=\frac{d^*|0\rangle-c^*|1\rangle}{\sqrt{|c|^2+|d|^2}}, \tag{7}$$

and

$$\left|y_2^-\right\rangle=\frac{c|0\rangle+d|1\rangle}{\sqrt{|c|^2+|d|^2}}. \tag{8}$$

Show that, although the first three conditions are satisfied, the constraint in Eq. (4) is violated.