
Exercise 3: Time evolution (3+2)

a) Consider a two-level system that evolves under the Hamiltonian $H = \hbar\omega\sigma_z$, with $\omega \in \mathbb{R}$. Let $\sigma_x, \sigma_y, \sigma_z$ denote the Pauli matrices. At time t_0 the system is in the state $|\psi(t_0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Calculate the state of the system at time $t \geq t_0$.

b) Calculate the expectation values of σ_x und σ_z at time t.

Exercise 4: Measurements and expectation values (3+2)

a) Calculate eigenvectors und eigenvalues of the following observable

$$A = \begin{pmatrix} \cos(\alpha) & -i\sin(\alpha) & 0 \\ i\sin(\alpha) & -\cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

by writing A in its spectral decomposition ($A = \sum_i \alpha_i |a_i\rangle\langle a_i|$ and $\langle a_i|a_j\rangle = \delta_{ij}$).

b) The observable A is being measured on a system in the state $|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$. Calculate the probabilities for the different outcomes and the expectation value of A.

Please turn!

Exercise 5: The tetrahedral POVM

The tetrahedral POVM is specified by the following unit vectors:

$$\begin{aligned}
\mathbf{n}_1 &= (0,0,1)^T, \\
\mathbf{n}_2 &= \left(\frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}\right)^T, \\
\mathbf{n}_3 &= \left(-\frac{\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, -\frac{1}{3}\right)^T, \\
\mathbf{n}_4 &= \left(-\frac{\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, -\frac{1}{3}\right)^T.
\end{aligned}$$

The POVM elements E_i $(i = 1, \ldots, 4)$ are given by $E_i = \frac{1}{4}(\mathbb{1} + \boldsymbol{n}_i \cdot \boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$.

a) Determine the effects E_i of the tetrahedral POVM.

b) Show that this is in fact a valid POVM.

c) Calculate the Naimark dilation of the tetrahedral POVM.