Foundations of Quantum Mechanics

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Sheet 0 — SS 2019 — Due: Wed., 10.04.2019

Exercise 1: Density matrices and Bloch sphere

a) Decide if the following matrices are quantum states. If so, determine if the state is pure or mixed.

(i)

$$\varrho_0 = \frac{1}{4}\begin{pmatrix} 1 & i \\ i & 3 \end{pmatrix}; \tag{1}$$

(ii)

$$\varrho_1 = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} - 2i \\ \frac{3}{2} + 2i & \frac{1}{2} \end{pmatrix}; \tag{2}$$

(iii)

$$\varrho_2 = \begin{pmatrix} \cos^2(\vartheta) & \cos(\vartheta)\sin(\vartheta)e^{-i\varphi} \\ \cos(\vartheta)\sin(\vartheta)e^{i\varphi} & \sin^2(\vartheta) \end{pmatrix} \tag{3}$$

where $\vartheta, \varphi \in \mathbb{R}$;

(iv)

$$\varrho_3 = \begin{pmatrix} \frac{5}{40} & -\frac{i}{100} \\ \frac{i}{100} & \frac{1}{16} \end{pmatrix}; \tag{4}$$

(v)

$$\varrho_4 = \frac{1}{2}\begin{pmatrix} 1 & (2p-1) \\ 2p-1 & 1 \end{pmatrix}, \tag{5}$$

where $0 \leq p \leq 2$.

b) Calculate the Bloch vector of the following density matrix

$$\varrho = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \tag{6}$$

where $a, c \in \mathbb{R}$, $a + c = 1$ and $\sqrt{(a-c)^2 + |b|^2} \leq 1$.

c) Show that for orthogonal states of a qubit $|\phi\rangle, |\psi\rangle$ the corresponding Bloch vectors $\vec{r}_\phi, \vec{r}_\psi$ admit the relation $\vec{r}_\phi = -\vec{r}_\psi$.

Please turn!

Exercise 2: Density matrices und Bloch sphere

a) Show that for any hermitian 2×2 matrix $A = (a_{ij})$ the following statements are equivalent:

   (i) A has no negative eigenvalues:

   (ii) $\langle\psi|A|\psi\rangle \geq 0$ for all $|\psi\rangle$;

   (iii) $\det(A) \geq 0$, $a_{11} \geq 0$ and $a_{22} \geq 0$.

b) Consider the qubit density matrix in the Bloch representation $\varrho = \frac{1}{2}(\mathbb{1} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)$. Determine the range of $\vec{r}$, for which the density matrix is positive semidefinite.