Statistical Physics
Assignment 8

Lecture: Prof. Dr. Otfried Gühne
Tutorial: Leonardo Novo, Tobias Moroder, Fri 8–10, Room: D115

Due to: Tue, 12.06

1. **Redlich-Kwong gas** (3 Points)
Consider a gas which satisfies the following equation of state
$$P = \frac{NT}{V-b} - \frac{a}{\sqrt{T}V(V+b)}, \tag{1}$$
with a, b being constants. Suppose that at low densities $V/N \ll b$ the specific heat is temperature independent, *i.e.*, $C_V(T) \approx const$. Use this information to compute the free energy $F(T, V)$.

2. **Barometric formula** (3 Points)
   (a) Consider an ideal gas in a generic potential $\Phi(\vec{x})$ and express the particle density $n(\vec{x}) = \langle \sum_i \delta(\vec{x}_i - \vec{x}) \rangle$ as a function of $\Phi(\vec{x})$.
   (b) Compute the particle density $n(\vec{x})$ for a linear potential of the form $\Phi(\vec{x}) = \vec{g} \cdot \vec{x}$, where $\vec{g}$ is a constant vector. Combine this with the equation of state for the ideal gas to finally express the pressure $P(\vec{x})$.

3. **Tonks gas** (5 Points)
Consider a one-dimensional system of N particles, each of which having a fixed length l. Those particles can occupy at most a length L (in analogy to a gas that can at most take a volume V). The particles cannot penetrate each other but otherwise there is no interaction between them. Compute the partition function and from that the free energy, the equation of state and the inner energy of this "gas".

   Hint: Note that the particles cannot change its order. In order to compute the partition function as a function of N it might help to consider the case $N = 2$ first, and then using induction.

4. **Stirling's formula** *
There are many ways to derive Stirling's approximation for $\log(n!)$. Here we take the following approach:
   (a) Via the Gamma function the factorial can expressed as
   $$n! = \int_0^\infty t^n \mathrm{e}^{-t} dt. \tag{2}$$
   Prove this identity, for instance by looking at $(-\partial/\partial\beta)^n \int_0^\infty \mathrm{e}^{-\beta t} dt \big|_{\beta=1}$.
   (b) Use a convenient substitution for t to show that
   $$n! = n^n \, \mathrm{e}^{-n} R_n, \text{ with } R_n = n \int_{-1}^\infty \mathrm{e}^{-n[t - \log(1+t)]} \, dt. \tag{3}$$
   (c) Decompose the integrand of R_n into a product of the form $\mathrm{e}^{-t^2 n/2} \mathrm{e}^{\cdots}$ and expand the second term up t^4 (Taylor expansion around $t = 0$). Then solve the integral by approximating it over the whole line $]-\infty, \infty[$.
   (d) The resulting terms of the Stirling's formula read
   $$\log(n!) = n \log n - n + \tfrac{1}{2} \log(2\pi n) + \mathcal{O}(n^x).$$
   Verify this and determine the parameter x.