Statistical Physics
Assignment 7

Lecture: Prof. Dr. Otfried Gühne
Tutorial: Leonardo Novo, Tobias Moroder, Fri 8–10, Room: D115

Due to: Tue, 05.06

1. **Equation of state and its consequences** (3 Points)
In thermodynamics an equation of state is a functional form connecting different (in- and extensive) state variables, *e.g.*, $f(x, y, z) = 0$ for variables x, y, z. This relation must be uniquely invertible such that variable $x = x(y, z)$ can for instance be uniquely expressed in terms of y, z. Similar relations must hold if one solves for any other variable. An example of such a state equation is $f(P, V, T) = P - Nk_BT/V$ for the ideal gas.

   For such an equation of state $f(x, y, z) = 0$ prove the following relations:

   (a)
$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{1}{\left(\frac{\partial y}{\partial x}\right)_z} \tag{1}$$

   (b)
$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1 \tag{2}$$

   Are they fulfilled for the ideal gas?

2. **Adiabatic state change** (6 Points)
Consider a system without heat exchange with the surrounding. Using the adiabatic exponent $\gamma = C_P/C_V$ one has the following differential equation

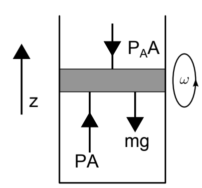
$$\left(\frac{\partial T}{\partial V}\right)_S = (1-\gamma)\left(\frac{\partial T}{\partial V}\right)_P, \tag{3}$$

   where C_P, C_V are the heat capacities at constant pressure and volume respectively. The partial derivative $\left(\frac{\partial T}{\partial V}\right)_P$ can typically be computed by an appropriate equation of state.

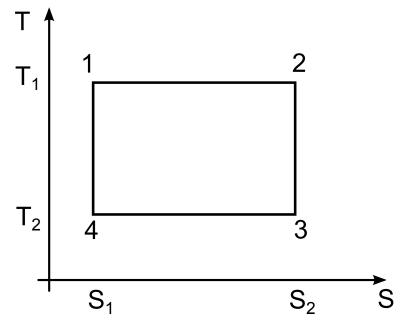
   (a) Show validity of this equation.

   (b) Solve this differential equation $T(V)$ for the case of an ideal gas, $pV = Nk_BT$.

   (c) Now let us see a possible measurement of the adiabatic exponent: An ideal gas in a container is held by a piston with mass m and area A. This piston acts onto the gas by its gravitational force mg and by the pressure of the atmosphere P_A, while the gas on the other hand acts by its pressure P. In the equilibrium state these two forces compensate exactly in which case the volume of the gas is V_0, from which the equilibrium pressure of the gas P_0 can be determined. The piston can however undergo a harmonic oscillation with frequency ω around this point of equilibrium; these oscillations can be assumed to be adiabatic. Formulate the equations of motions for the position of the piston z (using a first order approximation of $P(z)$ around the equilibrium position) in order to get the connection between the measurable quantities $\omega, m, V_0, A, P_A, g$ and the adiabatic exponent γ.



3. **Thermodynamic cycle** *
Consider the following reversible thermodynamic cycle of an ideal gas $PV = Nk_BT$ in the $T - S$ plane:



Here the process $1 \to 2$ and $3 \to 4$ are isotherm processes, while $2 \to 3$ and $4 \to 1$ are adiabatic and isentropic.

(a) Compute the heat change ΔQ for each process as a function of T_1, T_2 and S_1, S_2.

(b) How big is the work done by the system ΔW in one complete round and compute the efficiency $\eta = \frac{-\Delta W}{\Delta Q_1}$, where ΔQ_1 is the energy taken from the heat bath at temperature T_1 required at step $1 \to 2$.

(c) How does the process look in the corresponding $p - V$ plane?