Statistical Physics
Assignment 5

Lecture: Prof. Dr. Otfried Gühne
Tutorial: Leonardo Novo, Tobias Moroder, Fri 8–10, Room: D115

Due to: Tue, 15.05

1. **Ising interaction** (4 Points)
   Consider a linear chain of N spin-$\frac{1}{2}$ particles. Each spin can align parallel or anti-parallel in a magnetic field, *i.e.*,
   $$S_z^{(i)}|\sigma_1, \sigma_2, \ldots, \sigma_N\rangle = \sigma_i|\sigma_1, \sigma_2, \ldots, \sigma_N\rangle \tag{1}$$
   with $\sigma_i = \pm\frac{1}{2}$. A sudden arrangement of the spins can only occur if there is an interaction between the spins. In the following we like to consider the Ising-type interaction given by
   $$H = -\sum_{i=1}^{N-1} \gamma_i S_z^{(i)} S_z^{(i+1)}. \tag{2}$$
   (a) How many micro-states do we have if the number of spins is fixed to N? Express the partition function Z_N in terms of those micro-states.
   (b) Consider now the case that we extend the linear chain by one extra particle. Derive an recursion formula for the partition function $Z_{N+1} = f(Z_N)$.
   (c) Use this recursion together with $Z_1 = 2$ to obtain a final expression for Z_N.
   (d) Compute the correlation function $\left\langle S_z^{(i)} S_z^{(i+1)} \right\rangle$.

2. **Harmonic oscillator(s)** (6 Points)
   The quantum mechanical energy spectrum of a one-dimensional harmonic oscillator is given by
   $$E_\omega(n) = \hbar\omega\left(n + \frac{1}{2}\right) \tag{3}$$
   with frequency ω and $n = 0, 1, \ldots$. The energy eigenvalues are non-degenerate.
   (a) Compute the free energy as a function of the temperature.
   (b) How does the free energy behave in the limiting cases of high and low temperature?

   Now let us focus on the case of $3N$ different, independent harmonic oscillators. Then the corresponding eigenstates $|n_1, \ldots, n_{3N}\rangle$ with $n_i \in \mathbb{N}$ have energies
   $$E(n_1, \ldots n_{3N}) = \sum_{i=1}^{3N} E_{\omega_i}(n_i). \tag{4}$$
   (a) Compute the partition function under the Einstein assumption $\omega_j = \omega_E$ for all $j = 1, \ldots, 3N$.
   (b) Consider the heat capacity $C = \partial_T U$ and discuss the result for the case $T \ll T_E$ and $T \gg T_E$, with T_E being the Einstein temperature $T_E = \beta\hbar\omega_E$. Compare this with the classical results known as the Dulong-Petit law $C_{\text{classical}} = 3Nk_B$.

3. **Lattice gas** *
   Consider the scenario that N distinguishable particles are distributed over a lattice. This lattice contains N ground and N intermediate positions, each of which can be occupied with at most one particle. Whenever a particle takes a ground state it costs no energy, however an intermediate position requires energy ε.

(a) How large is the energy if M intermediate positions are occupied in equilibrium?

(b) Compute the entropy of the equilibrium state with exactly M intermediate positions being filled and consider the limit of $N, M \gg 1$ (Hint: Stirling's formula).

(c) Investigate whether this limiting entropy is an extensive (additive) property. Extensive means that if the system is subdivided into two parts, that the sum of both individual entropies equals the total entropy.

(d) Bonus: What happens if the N are indistinguishable; is the entropy extensive?