Statistical Physics
Assignment 3

Lecture: Prof. Dr. Otfried Gühne
Tutorial: Leonardo Novo, Tobias Moroder, Fri 8–10, Room: D115

Due to: Wed, 02.05, Return room: B111

1. **Operator calculus** (4 Points)
   (a) Describe all Hermitian matrices A with $\delta\,\mathrm{tr}(A^3 - 3A) = 0$.
   (b) Find a non-Hermitian matrix which satisfies
   $$\delta\,\mathrm{tr}(AA^\dagger A - 3A) = 0. \tag{1}$$
   (Hint: For $i \neq j$ it holds for instance $(\sigma_i + i\,\sigma_j)^2 = 0$.)
   (c) Determine X according to the series expansion
   $$\exp[\sigma_x + \varepsilon\sigma_z] = \exp[\sigma_x] + \varepsilon\,X + \mathcal{O}(\varepsilon^2), \tag{2}$$
   with σ_x, σ_z being the corresponding Pauli matrices.

2. **Cumulants** (3 Points)
   Which probability density $f(x)$ has the following cumulants
   $$\langle\langle X\rangle\rangle = 2\sigma,\ \langle\langle X^2\rangle\rangle = \lambda,\ \langle\langle X^k\rangle\rangle = 0,\ \forall k \geq 2? \tag{3}$$

3. **Entropy***
   The quantum relative entropy $S(\varrho\|\sigma)$ is defined as
   $$S(\varrho\|\sigma) = \mathrm{tr}[\varrho\log\varrho - \varrho\log\sigma]. \tag{4}$$
   As in the classical case it satisfies $S(\varrho\|\sigma) \geq 0$ for any pair of quantum states ϱ, σ.
   (a) Show that $S(\varrho\|\sigma)$ equals the classical relative entropy of the eigenvalues of ϱ and σ if both of them are diagonal in the same basis.
   (b) Compute the relative entropy for $S(\varrho_{\mathrm{AB}}\|\varrho_{\mathrm{A}} \otimes \varrho_{\mathrm{B}})$ to prove the sub-additivity relation
   $$S(\varrho_{\mathrm{AB}}) \leq S(\varrho_{\mathrm{A}}) + S(\varrho_{\mathrm{B}}). \tag{5}$$
   (c) Finally employ sub-additivity on $\varrho_{\mathrm{AB}} = \sum_i p_i \varrho_{i,\mathrm{A}} \otimes |i\rangle_{\mathrm{B}}\langle i|$ to show concavity of the von Neumann entropy $\sum_i p_i S(\varrho_i) \leq S(\sum_i p_i \varrho_i)$.

4. **Information content** (3 Points)
   Compute the single letter entropies of the languages English, French and German based on the following relative letter appearances to decide which language is more random, *i.e.*, contains more information per letter.

   For simplicity consider the case that the letter 'Others' is one more extra letter. (Be aware that the table is not normalized!).

Letter	English	French	German
a	0.08	0.08	0.07
b	0.01	0.01	0.02
c	0.03	0.03	0.03
d	0.04	0.04	0.05
e	0.13	0.15	0.17
f	0.02	0.01	0.02
g	0.02	0.01	0.03
h	0.06	0.01	0.05
i	0.07	0.08	0.08
j	0.00	0.01	0.00
k	0.01	0.00	0.01
l	0.04	0.05	0.03
m	0.02	0.03	0.03
n	0.07	0.07	0.10
o	0.08	0.05	0.03
p	0.02	0.03	0.01
q	0.00	0.01	0.00
r	0.06	0.07	0.07
s	0.06	0.08	0.07
t	0.09	0.07	0.06
u	0.03	0.06	0.04
v	0.01	0.02	0.01
w	0.02	0.00	0.02
x	0.00	0.00	0.00
y	0.02	0.00	0.00
z	0.00	0.00	0.01
Others	0	0.03	0

Table 1: Average frequencies of the 26 most common Latin letters, taken from `wiki.stat.ucla.edu/socr/index.php/SOCR_LetterFrequencyData`.