Statistical Physics
Assignment 2

Lecture: Prof. Dr. Otfried Gühne
Tutorial: Leonardo Novo, Tobias Moroder, Fri 8–10, Room: D115

Due to: Tue, 24. 04

1. **Boar hunt** (3 Points)
Three hunters are shooting on an escaping boar. Hunter 1 shoots three times as often and hunter 2 twice as often as hunter 3. After a while the boar is dead and the hunters have to decide how has been the lucky one who finally caught it. For that the hunters base their decision on their aiming accuracy at the firing range, which are $0.3, 0.6$ and 0.8 respectively. Compute the conditional probabilities that the deadly bullet came from hunter k.

2. **Triel** (5 Points)
Unfortunately the three hunter of the previous example couldn't really decide on who really gets the trophy, so they challenge each other for a "triel" (duel with three participants). However before that each of them is practising once more at the firing range to improve their accuracy to $0.5, 0.8$ and 1. The triel starts with the weakest shooter first, followed by the second strongest and then the best shooter, and then repeating this procedure. Each hunter has enough time to aim and shoot at the desired target. Compute the survival probabilities for each hunter depending on the following strategies:

   (a) Each one shoots at the strongest enemy.

   (b) Same as (a), but considering now that the weakest one shoots in the air while he's still facing two enemies.

3. **Berkeley gender bias case** (4 Points)
The University of California, Berkeley was sued for bias against women who had applied for admission to graduate schools. The admission figures for the fall of 1973 showed the following behaviour:

	Applicants	Admitted
Men	8442	44 %
Women	4321	35 %

   (a) We want to know whether this sue is really justified because these fluctuations can also be due to bad luck. For that consider the model that the acceptance probability for a female applicant is p_{women}, and that each applicant is decided upon independently. Use Hoeffding's inequality to compute a bound on the probability that a relative admission fraction of at most 40% is present in a sample of 4321 woman if $p_{\text{women}} \geq 0.44$.

   (b) However the above model is misleading: A more appropriate and fair comparison would be an admission probability per department. Explain the above data by a scenario with just two departments, *e.g.*, only physics and art, but where the conditional acceptance probabilities for women are always larger than for men. (Hint: Maybe more women apply for physics than for art.)

4. **Hoeffding's tail inequality***
In this exercise we want to derive Hoeffding's tail inequality. This is approached in several steps:

   (a) Consider a (real-valued) non-negative random variable X which takes only values from a finite alphabet, x_i with $i = 1, \ldots, N$. First let us prove the following inequality

$$Prob[X \geq t] \leq \frac{\mathbb{E}(X)}{t}, \tag{1}$$

where $\mathbb{E}(X) = \sum_i P(x_i)x_i$ stands for the mean value. (Hint: The inequality can be derived starting from this mean value expression and distinguishing the parts $x_i < t$ and $x_i \geq t$.)

(b) Use the inequality given by Eq. (**??**) to prove that a general real-valued random variable X satisfies

$$Prob[X \geq t] \leq \frac{\mathbb{E}[\phi(X)]}{\phi(t)} \tag{2}$$

for any strictly monotonically increasing non-negative valued function ϕ.

(c) Consider the sum of independent random variables X_i, *i.e.*, $\bar{X} = 1/N \sum_i X_i$. Using independence and $\phi(\bar{X}) = e^{s\bar{X}}$ to prove

$$Prob[\bar{X} \geq t] \leq \min_s \left\{ e^{-stN} \prod_i \mathbb{E}[e^{sX_i}] \right\}. \tag{3}$$

(d) Finally, using Eq. (**??**) together with the bound $\mathbb{E}[e^{sX_i}] = e^{s^2(b_i-a_i)^2/8}$, which holds for X_i with $\mathbb{E}(X_i) = 0$ and $a_i \leq X \leq b_i$, one arrives at Hoeffding's inequality,

$$Prob[\bar{X} \geq t] \leq e^{-2t^2N^2/\sum_i(b_i-a_i)^2}. \tag{4}$$