Statistical Physics
Assignment 10

Lecture: Prof. Dr. Otfried Gühne
Tutorial: Leonardo Novo, Tobias Moroder, Fri 8–10, Room: D115

Due to: Tue, 26.06

1. **Particle density** (4 Points)
   We consider an ideal quantum gas in d dimensions. The density $\varrho(\epsilon)$ is defined such that $\varrho(\epsilon)d\epsilon$ gives the number of states with energies between ϵ and $\epsilon + d\epsilon$.
   (a) Show that the surface of the d-dimensional unit sphere is given by
   $$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \tag{1}$$
   where $\Gamma(x) = \int_0^\infty dt\, t^{x-1}e^{-t}$ is the Gamma function. Hint: Integrate $e^{-|\vec{r}|^2}$ in Cartesian and spherical coordinates.
   (b) The dispersion relation, *i.e.*, the connection between energy and momentum of a particle, is given in the form $\epsilon = \epsilon(p)$, with $\epsilon(p)$ being an increasing function. Here $p = |\vec{p}|$ is the absolute value of the d-dimensional momentum. Compute the density $\varrho(\epsilon)$.
   (c) Explicitly calculate the density for the quadratic dispersion relation $\epsilon(p) = p^2/(2m)$ in the dimensions $d = 1, 2, 3$. Plot this density, for the three cases.

2. **Two-dimensional quantum gas** (5 Points)
   We consider an ideal Bose- or Fermi gas in two dimensions. As derived in the previous exercise the density is given by
   $$\frac{A}{(2\pi)^2}\sum_{m_s}\int d^2k f(\epsilon_k) = \int d\epsilon \varrho(\epsilon) f(\epsilon) \text{ with } \varrho(\epsilon) = \frac{gAm}{2\pi\hbar^2}. \tag{2}$$
   Here A is the corresponding area and the factor $g = 2s + 1$ accounts for the spin degeneracy. In the following consider bosonic and fermionic case:
   (a) Compute the chemical potential as a function of the particle density N/A and the temperature T. Draw a sketch of μ as a function of T.
   (b) Recall the virial expansion
   $$P = \frac{N}{A}k_BT\left[1 + B_1(T)\frac{N}{A} + \mathcal{O}(\frac{N}{A})^2\right]. \tag{3}$$
   Compute the virial coefficient $B_1(T)$ by using $P = -\partial_A J(T, A, \mu)$ together with an expansion in the density N/A.

3. **Jordan-Wigner transformation** *
   Even spin systems can be described with appropriate creation and annihilation operators. In this part we focus on the spin-$\frac{1}{2}$ case: Let S_j^α denote the spin operator of j-th particle in direction $\alpha = x, y, z$. We have the following commutation relations
   $$\left[S_j^\alpha, S_k^\beta\right] = i\delta_{jk}\sum_\gamma \epsilon_{\alpha\beta\gamma}S_j^\gamma. \tag{4}$$
   Let us now define
   $$S_j^\pm = S_j^x \pm iS_j^y \text{ and } c_k = \Big[\prod_{j=1}^{k-1}(2S_j^+S_j^- - \mathbb{1})\Big]S_k^+. \tag{5}$$
   (a) Verify that c_k defines a valid annihilation operator.
   (b) How does the vacuum state look like?