Statistical Physics
Assignment 1

Lecture: Prof. Dr. Otfried Gühne
Tutorial: Leonardo Novo, Tobias Moroder, Fri 8–10, Room: D115

Due to: Tue, 17.04

1. **Density matrices** (4 Points)
Consider the following family of matrices:

$$\varrho_{\mathrm{AB}}(x_1, x_2, x_3) = \frac{1}{17} \begin{pmatrix} 5 & 0 & 0 & x_2 \\ 0 & x_1 & -x_3/127 & 0 \\ 0 & 1 & 3 & 0 \\ x_2 & 0 & 0 & 7 \end{pmatrix}. \tag{1}$$

   (a) For which parameters $x_i \in \mathbb{C}$ does $\varrho_{\mathrm{AB}}(x_1, x_2, x_3)$ represent a density operators?

   (b) Compute the reduced matrices $\varrho_A(x_1, x_2, x_3)$, $\varrho_B(x_1, x_2, x_3)$ and re-examine which parameters $x_i \in \mathbb{C}$ form valid density operators.

2. **Superposition and mixture** (3 Points)
Consider the density matrix $\varrho = |\psi^-\rangle\langle\psi^-|$ of the singlet state ($|\psi^-\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$) and the density matrix $\varrho' = (|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|)/2$ of the statistical mixture of $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$.

   (a) Show that there exists no unitary U which satisfies $\varrho = U\varrho' U^\dagger$.

   (b) Can one discriminate between ϱ and ϱ' based on one of the expectation values of $\sigma_x \otimes \sigma_x$, $\sigma_y \otimes \sigma_y$ or $\sigma_z \otimes \sigma_z$?

3. **Decomposition of density matrices***
A density matrix ϱ with $rank(\varrho) = n$ given in its spectral decomposition

$$\varrho = \sum_{i=1}^{n} \lambda_i |e_i\rangle\langle e_i|, \tag{2}$$

can be interpreted as the statistical mixture of pure states $|e_i\rangle$ according to probabilities λ_i. However, the same density operator ϱ can also be interpreted as the mixture of other pure states $|\psi_i\rangle$ and different probabilities p_i if and only if it holds

$$\sqrt{p_i}|\psi_i\rangle = \sum_{j=1}^{n} U_{ij} \sqrt{\lambda_j} |e_j\rangle \tag{3}$$

with U being a unitary.

   (a) Show that if Eq. (3) holds and U is unitary then ϱ as given in Eq. (2) can also be written as $\varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

   (b) Equation (3) can be employed to prove that there always exists an ensemble with uniform distribution, *i.e.*, $p_i = \frac{1}{n}$. Prove this statement explicitly for the state

$$\varrho = \lambda|\uparrow\rangle\langle\uparrow| + (1-\lambda)|\downarrow\rangle\langle\downarrow| \tag{4}$$

   with $\lambda \in (0, 1)$. That means find the corresponding states $|\psi_i\rangle$ such that

$$\varrho = \frac{1}{2} \left(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| \right) \tag{5}$$

   (Remark: U can be taken to be real and $\det(U) = 1$ in this case.)