# Quantum Information Theory
# Exercise sheet 2

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Lecture: Tuesday, 10-12, Room D 120
Exercise: Monday, 15-17, Room B 107

**5. Tsirelson bound**
We want to derive a bound on the maximal violation of the CHSH inequality allowed by quantum mechanics.

(a) Verify that the square of the CHSH operator $B = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2$ (where $A_i^2 = B_i^2 = \mathbf{1}$ are any dichotomic observables) is given by

$$B^2 = 4\mathbf{1} - [A_1, A_2] \otimes [B_1, B_2].$$

(b) Convince yourself that for all observables $C$ and $D$:

- $\langle C \rangle^2 \leq \langle C^2 \rangle$
- $\max_\rho \operatorname{Tr}(C\rho) = \max_\psi \langle \psi | C | \psi \rangle$, that is, the maximum is assumed at a pure state.
- $\max_\psi \big| \langle \psi | C \otimes D | \psi \rangle \big| = \max_{\phi_A} \big| \langle \phi_A | C | \phi_A \rangle \big| \cdot \max_{\phi_B} \big| \langle \phi_B | D | \phi_B \rangle \big|$, that is, the maximum is assumed at a product state.

(c) Use these results to prove the Tsirelson bound $|\langle B \rangle| \leq 2\sqrt{2}$.

**6. Nonlocal box**
We consider a theory that assigns the following probabilities to the measurement results for the observables in a CHSH inequality:

| | | $A_1$ | | $A_2$ | |
|---|---|---|---|---|---|
| | | $+1$ | $-1$ | $+1$ | $-1$ |
| $B_1$ | $+1$ | $1/2$ | $0$ | $1/2$ | $0$ |
| | $-1$ | $0$ | $1/2$ | $0$ | $1/2$ |
| $B_2$ | $+1$ | $1/2$ | $0$ | $0$ | $1/2$ |
| | $-1$ | $0$ | $1/2$ | $1/2$ | $0$ |

For example, when measuring $A_1$ on the first system and $B_2$ on the second system simultaneously, the probability of obtaining the result $+1$ for $A_1$ and $+1$ for $B_2$ is $1/2$, while the probability of obtaining $+1$ and $-1$ is zero.

(a) Verify that this theory is non-signalling, i. e., it does not allow faster-than-light communication.

(b) Show that within this theory

$$\langle A_1 \otimes B_1 \rangle + \langle A_1 \otimes B_2 \rangle + \langle A_2 \otimes B_1 \rangle - \langle A_2 \otimes B_2 \rangle = 4.$$

(c) Which of the conditions leading to the local hidden variable bound of 2 is not fulfilled by this theory? Can the probability table considered here originate from a quantum state? Can the Tsirelson bound be derived from the non-signalling principle?