Quantum Information Theory
Exercise sheet 1

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Lecture: Tuesday, 10-12, Room D 120
Exercise: Monday, 15-17, Room B 107

1. **Positive semidefinite matrices**
   Show that for hermitean 2×2-matrices $A = (a_{ij})$ the following statements are equivalent:
   - A has no negative eigenvalues;
   - $\langle\psi|A|\psi\rangle \geq 0$ for all vectors $|\psi\rangle$;
   - $\det A \geq 0$, $a_{11} \geq 0$ and $a_{22} \geq 0$.

2. **Bloch vectors**
   Any single-qubit density matrix ρ can be parameterized in terms of the Pauli matrices by a Bloch vector $\vec{a}$:
   $$\rho = \frac{1}{2}\big(\mathbf{1} + \sum_{i=1}^{3} a_i \sigma_i\big).$$
   On the other hand, ρ can be expressed in its eigenbasis as
   $$\rho = p|\phi_1\rangle\langle\phi_1| + (1-p)|\phi_2\rangle\langle\phi_2|$$
   with $0 \leq p \leq 1$.
   (a) Determine $\det\rho$. Which condition on $\vec{a}$ results from $0 \leq p \leq 1$? Describe the set of all density matrices geometrically.
   (b) Describe a minimal set of observables, such that from the corresponding mean values $\vec{a}$ and therefore ρ can be reconstructed.
   (c) Determine p as a function of $\vec{a}$, and compute the eigenvectors $|\phi_i\rangle$ as a function of $\vec{a}$.

3. **Schmidt decomposition**
   Determine the Schmidt decomposition (in (c) and (d) for all bipartitions) of the following states (this can be done without long computation):
   (a) $(|00\rangle + |01\rangle + |10\rangle - |11\rangle)/2$
   (b) $(|00\rangle + |01\rangle + |10\rangle + |11\rangle)/2$
   (c) $(|000\rangle + 2|011\rangle + |110\rangle)/\sqrt{6}$
   (d) $(|000\rangle + \mathrm{i}|010\rangle + \mathrm{i}|101\rangle - |111\rangle)/2$

4. **GHZ- and W state**
   In this excercise we want to investigate two three-qubit states with different entanglement properties: the GHZ state $|\mathrm{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ and the W state $|\mathrm{W}\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$.
   (a) Determine the reduced two-qubit states. Are they entangled or separable? (Hint: It is known that a two-qubit state ρ is entangled, if its *fidelity* with a Bell state $|\mathrm{BS}\rangle$ exceeds $1/2$, that is, if $\langle\mathrm{BS}|\rho|\mathrm{BS}\rangle > 1/2$.)
   (b) Which states arise from $|\mathrm{GHZ}\rangle$ and $|\mathrm{W}\rangle$, if σ_z is measured on a single qubit? Note that this depends on the measurement result. Which states arise if σ_x is measured?