The Kadison-Singer Conjecture
-
A survey

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Abstract

The Kadison-Singer problem is a question in operator theory and was posed 1959 by R. Kadison and I. Singer. It asks whether every pure state on a maximal abelian subalgebra of the algebra of bounded linear operators $B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ has a unique pure extension to a pure state on $B(\mathcal{H})$. Between the original formulation and its solution in 2013 many equivalent problems were formulated, settled in vastly different areas of mathematics such as discrepancy theory, graph theory and operator theory. We present the mathematical foundations necessary in order to formulate the Kadison-Singer problem, its equivalent formulations as well as its solution. Further, we discuss the particular formulation of the problem, how Kadison and Singer exclude the case of continuous maximal abelian subalgebras and the equivalent formulations in the form of Anderson's paving conjecture and Weaver's conjecture. Thereupon, following Marcus, Spielman and Srivastava, we present the proof of Weaver's conjecture implying a positive solution to the Kadison-Singer problem. Finally, we discuss the consequences of the result for the existence of infinite families of Ramanujan graphs of any degree as well as for problems in harmonic analysis.

Contents

Chapter 1

Introduction

The Red-Haired Man

There was a red-haired man who had no eyes or ears.
Neither did he have any hair, so he was called red-haired by convention.
He couldn't speak, since he didn't have a mouth. Neither did he have a nose.
He didn't even have any arms or legs. He had no stomach and he had no
back and he had no spine and he had no innards whatsoever.
He had nothing at all!
Therefore there's no knowing whom we are even talking about.
In fact it's better that we don't say any more about him.

Daniil Charms

The exploitation of new mathematical fields is intimately connected to the discovery of new physical theories, or more precise, to the attempts of describing observed events within a mathematical framework [1]. Most popular examples include the correspondence between calculus and classical mechanics, electrodynamics and Fourier-Analysis as well as general relativity and differential geometry. A similar fruitful interplay between mathematics and physics can also be observed in the case of quantum mechanics and C^*-algebras. Here, C^*-algebras were considered for their use to model algebras of physical observables. In particular, the line of research began with Heisenberg's matrix mechanics and in a more mathematically developed form with the work of Jordan around 1933 [2, 3]. Afterwards, von Neumann established a general framework for these algebras, which culminated in a series of papers on rings of operators [4, 5, 6] as well as his influential book on quantum mechanics [7].

At the same time, also Dirac attempted to compose a complete mathematical treatise of quantum mechanics with an additional focus on the quantization of the electromagnetic field. In difference to von Neumann, Dirac, strongly influenced from the Kopenhagen interpretation, formulated the theory in terms of Hilbert space vectors. In particular, one associates to a physical system a Hilbert space, where the state of the system is described by a unit vector and dynamical variables or observables correspond to self-adjoint operators. It is important to note, that this formalism is only capable of dealing with pure

Daniil Charms, born 1905 in Petersburg as Daniil Iwanowitsch Juwatschow was an early Soviet-era avant-gardist and absurdist poet, writer and dramatist. He was member of the artistic circle Oberiu, known for its futuristic art. He died 1942 under arrest, probably during the siege of Leningrad.

states, while the von Neumann formalism includes mixed states in a natural way, namely as positive, normalized linear functionals on the algebra of observables [8, 9]. More precisely, taking convex mixtures of states has the operational interpretation of randomization in physics. While pure states are seen as states of maximal knowledge, so called ontic, mixed states include uncertainties in the preparation, so called epistemic states [1]. Due to the measurement postulate of quantum mechanics, one cannot in general assign a dispersion free value simultaneously to multiple observables. But one can give a meaning to several commuting observables having values at the same time. In this context, the notion of a complete set of commuting observables plays an important role. In order to characterize a quantum state uniquely, it is often necessary to consider multiple observables, e.g., for the hydrogen atom it is not sufficient to only specify the energy E, but also the absolute of the angular momentum l and its z-component m, such that a state is uniquely characterized by $|E, l, m\rangle$. Therefore, Dirac wanted to find representations, that are orthonormal bases, for a compatible family of obervables, that is a commuting family of self-adjoint observables. He states [17]:

> To introduce a representation in practice:
>
> (i) We look for observables which we would like to have diagonal, either because we are interested in their probabilities or for reasons of mathematical simplicity;
>
> (ii) We must see that they all commute- a necessary condition since diagonal matrices always commute;
>
> (iii) We then see that they form a complete commuting set, and if not, we add some more commuting observables to them to make them into a complete commuting set;
>
> (iv) We set up an orthogonal representation with this complete commuting set diagonal.
>
> The representation is then completely determined by except for the arbitrary phase factors. For most purposes the arbitrary phase factors are unimportant and trivial, so that we may count the representation as being completely determined by the observables that are diagonal in it.

In a mathematically precise form, Dirac claims that each pure state on a complete commuting set has a unique extension to $B(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space. If the Hilbert space is finite dimensional, i.e., $\mathcal{H} = \ell_2(I)$ with $|I| = n \in \mathbb{N}$, this statement is unsuspicious, since up to isomorphisms there is only one maximal abelian subalgebra. However, in general there are more. For instance, the so called discrete maximal abelian subalgebra of $B(\ell_2(\mathbb{N}))$, which can be identified with $\ell^\infty(\mathbb{N})$, or the continuous maximal abelian subalgebra of $B(\ell_2(\mathbb{N}))$, identified with $L^\infty([0,1])$. An interesting peculiarity occurs if one considers observables where the spectrum is given by closed intervals, a special instance of operators with a continuous spectrum. In fact, this problem led Dirac to introduce the famous δ-function, in order to resolve the problem of normalization.

[1]The realization that the state space of quantum mechanics is given by a convex set of positive linear functionals led to a further abstraction, which is nowadays called generalized probabilistic theories. Here, any compact convex set can serve as the state space of a theory, hence comprising classical and quantum theory but also vastly different ones. For instance, the so called Popescu-Rohrlich box [10], spin factors [11] and theories corresponding to Jordan algebras [12]. For a more detailed description see [13, 14, 15, 16].

The uniqueness of extension claimed by Dirac was questioned by Kadison and Singer and led to the famous Kadison-Singer problem as well as to their seminal paper [18] in 1959. It was already known at this time, that Dirac's claim holds for vector pure states, that is, states ω_x given by $\omega_x(A) = \langle Ax, x\rangle$, where $x \in \mathcal{H}$ unit vector. According to quantum theory, $\omega_x(A)$ is the expectation value of the observable A if the physical system is in state x. Apart from this type of states there are much more pure states on $\ell^\infty(\mathbb{N})$. More precisely, the set of all pure states on $\ell^\infty(\mathbb{N})$ equipped with the w*-topology can be identified with $\beta(\mathbb{N})$, the Stone-Čech compactification of $\mathbb{N}$. However, whether these kind of states still have a physical interpretation can be questioned from an operational perspective. In particular, the preparation of states whose description involves the axiom of choice, hence a highly non constructive component, seems problematic. Therefore the question of uniqueness of extension is more about a certain mathematical apparatus around quantum mechanics, rather about the physics itself.

Kadison and Singer showed [18], that for the continuous maximal abelian subalgebra of $B(\ell_2(\mathbb{N}))$, which can be identified with $L^\infty([0,1])$ via multiplication operators (cf. Example 2.48), the extension of states is not unique. Their proof relies on so called diagonal processes, an operator theoretic generalization of taking the diagonal of a matrix, introduced by von Neumann in 1940. Further they proved for the discrete case, i.e., where the commutative subalgebra is identified with $\ell^\infty(\mathbb{N})$, that each pure state of $\ell^\infty(\mathbb{N})$ has a unique extension to the norm closure of the algebra of linear operators T_π defined by $T_\pi e_i = e_{\pi(i)}$, where π is a permutation of $\mathbb{N}$. Though a lot of efforts, they were neither able to prove the uniqueness of state extension nor to construct an explicit counterexample as for the continuous case. However, they state that they incline to the view that the extension is non unique. A more detailed description of those states which have the unique extension property was given 1970 by Reid [19]. By using recent results about ultrafilters, which were developed in 1968 by Choquet [20, 21], he proved that those pure states corresponding to so called rare ultrafilter have a unique extension to $B(\ell_2(\mathbb{N}))$. Unfortunately, it was unknown whether or not all the points of $\beta(\mathbb{N}) \setminus \mathbb{N}$ are in correspondence to rare ultrafilters, hence not providing a solution to the problem. A new impetus towards a solution was given in 1979 by Anderson [22], who approached the problem from a more general perspective. Namely, he starts with an arbitrary subalgebra $B \subset A$ of an C^*-algebra A and asks for the characterization of those pure states on B, which uniquely extend to A. To do so, he introduced the property of an C^*-algebra A to be B-compressible modulo φ, where B is a subalgebra and φ a state on B. With this at hand, he could give a complete characterization of pure states on B which extend uniquely to A. In particular, only states φ on B for which A is B-compressible modulo φ extend uniquely. The question whether $B(\ell_2(\mathbb{N}))$ is $\ell^\infty(\mathbb{N})$-compressible for any pure state on $\ell^\infty(\mathbb{N})$ became known as Anderson's infinite paving conjecture and was also reformulated in a finite dimensional version [23].

In the following years many different equivalent versions of the Kadison-Singer problem were found, mainly by Feichtinger, Paulsen, Akemann, Bourgain and Tzafriri [24, 25, 26]. However, the most important equivalent formulation was found in 2002 by Weaver, who related the problem to a question in discrepancy theory, i.e., the branch of mathematics that seeks to understand how well a continuous object can be approximated by a discrete one. Weaver's approach is based on the Anderson paving reduction but using orthogonal projections with near-zero diagonal.

Finally, in 2013 the conjecture of Weaver was proven by Marcus, Spielman and Srivastava [27] using a technique of random polynomials and thus gave an affirmative answer to the Kadison-Singer problem. Using the probabilistic method, first introduced by Erdös in 1959, the crucial point is to realize that showing that a random matrix has all small eigenvalues with nonzero probability is a special case of the more generic problem of showing that some polynomial from a collection must have all small roots. To achieve this, they develop the method of interlacing families, a device that allows one to draw the desired conclusion by studying the roots of the average of the polynomials in such a collection.

That this method is indeed powerful has also turned out, when it was applied to graph theory [28]. In 1988 Lubotzky, Phillips and Sarnak [29] defined so called Ramanujan graphs to be d-regular graphs, where the largest nontrivial eigenvalue is bounded by $2\sqrt{d-1}$, which is independent of the number of vertices. The importance of these graphs comes from the fact, that they are optimal in the sense that the spectrum of their adjacency matrix is contained in the smallest possible interval. Indeed, by the Alon-Boppana bound [30, 31] for d-regular graphs, for an $\epsilon > 0$ given, one can always find a $n \in \mathbb{N}$ such that all d-regular graphs on n vertices have one eigenvalue at least $2\sqrt{d-1} - \epsilon$. While the construction of Ramanujan graphs of degree $d \in \mathbb{N}$ is easy for a small number of vertices, it is difficult as the number of vertices grows. In particular, it was a long standing problem whether there exist infinite families of Ramanujan graphs for all degrees. Obviously, one option would be to start with a small d-regular graph which is Ramanujan and then, via an appropriate duplication procedure, create new larger d-regular graphs in such a way, that no new eigenvalues larger then $2\sqrt{d-1}$ are introduced. This idea is formalized by the notion of 2-lifts, where one assigns to each edge an element of $\{\pm 1\}$, and construct from this a graph with twice the vertices. The totality of the sign assignments is called signature. That this procedure works indeed, is the content of a conjecture of Bilu and Linial [32]. Marcus, Spielman and Srivastava proved a weak version of this conjecture which applies to bipartite graphs, namely that every d-regular graph has a signing in which all of the new eigenvalues are at most $2\sqrt{d-1}$.

The aim of this thesis is to give a systematic introduction into the abstract mathematical framework needed in order to formulate the Kadison-Singer problem, its equivalent statements and its solution. For this purpose Chapter 2 introduces topological notions such as compactification of a topological space, filters and nets. In addition, also local convex topological spaces are introduced via a family of seminorms. We then proceed by recapitulate the basic facts from the theory of C^*-algebras, including an analysis of the structure of the state space as well as the classification of C^*-algebras based on the GNS-construction. Subsequent, we focus on von Neumann algebras and its characterization by von Neumann's double commutant theorem. Further we present a classification of the maximal abelian $*$-algebras in $B(\mathcal{H})$. In Section 3 we first introduce the Kadison-Singer problem in the form posed in [18]. We then ask the question why this particular formulation of the problem was chosen. More precise, we start with a more general question and processing the question using the content of Section 2 in order to obtain the Kadison-Singer problem. We proceed by deriving the most popular equivalent formulations of the Kadison-Singer problem, namely the Anderson's finite- and infinite paving conjecture as well as Weaver's conjecture. Section 4 deals with the proof of an affirmative answer of the Kadison-Singer problem. In particular, following the work of Marcus, Spielman and Srivastava [27], we introduce the concepts of interlacing families, real stable polynomials as well as the so called multivariate barrier argument to prove Weaver's conjecture. Concluding with Section 5, we present how the machinery used in the proof of Weaver's conjecture could also be utilized in order to resolve a long standing problem in spectral graph theory, namely the existence of infinite families of Ramanujan graphs of any degree.

Chapter 2

The Language of the Problem

Before we are able to formulate the Kadison-Singer problem in Section 3 it is necessary to introduce the language within this problem is formulated. First, we will revisit the topological aspects such as compactifications, filters and nets. We will then proceed to consider the operator algebraic formalism, where the notion of C^*- and von Neumann algebras and states thereof play a crucial role. Finally, we turn to so called frames, that can be seen as an overcomplete basis, where the generating vectors still obey certain geometrical relations. In particular, Parseval frames are a key ingredient in order to formulate and solve Weaver's conjecture which implies a positive solution to the Kadison-Singer problem. For a complete treatise on topological spaces we refer to [6, 33, 34], while a classical course on operator algebras are [35, 36, 37, 38]. A more modern and recent approach is given in [39, 40]. An introduction to frames and their applications can be found in [41].

2.1 Compactifications, filters and nets

Definition 2.1. Let $(X, \mathcal{T})$ be a topological space[1].

(a) $(X, \mathcal{T})$ is a T_2-space if points $x, y \in X$ can be separated by neighbourhoods, i.e., one can find neighbourhoods $N_x, N_y \subset X$ with $x \in N_x$ and $y \in N_y$ such that $N_x \cap N_y = \emptyset$

(b) $(X, \mathcal{T})$ is called regular, if for any closed $A \subset X$ and any $x \in X$ with $x \in X \setminus A$ there exist neighbourhoods $N_x, N_A \subset X$ such that $x \in N_x$, $A \subset N_A$ and $N_x \cap N_A = \emptyset$. A regular T_2-space is called a T_3-space.

(c) $(X, \mathcal{T})$ is called completely regular, if for any closed set $A \subset X$ and any point $x \in X \setminus A$ there exists a real-valued continuous function $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f|_A = 0$. A completely regular T_2-space is called a $T_{3\frac{1}{2}}$-space or Tychonoff-space.

(d) $(X, \mathcal{T})$ is called a normal space, if given two disjoint closed subsets $A, B \subset X$ there are neighbourhoods $N_A, N_B \subset X$ such that $A \subset N_A$, $B \subset N_B$ such that $N_A \cap N_B = \emptyset$. A normal T_2-space is called a T_4-space.

[1] This list of separation axioms of topological spaces is just a collection of special properties needed to develop the concept of Stone- Čech compactifications. Except the four different axioms listed here, there are at least four more different separation axioms, namely T_1, $T_{2\frac{1}{2}}, T_5$ and T_6. For a more detailed account of separation axioms in topological spaces see [33].

Definition 2.2. Let $(X, \mathcal{T})$ be a topological space. X is called compact, if each of its open covers has a finite subcover, i.e., for every collection $(C_\lambda)_{\lambda \in \Lambda}$ of open subsets of X with $\Lambda \neq \emptyset$ index set, there exists a finite subset $I \subset \Lambda$ such that $X = \cup_{i \in I} C_i$. A subset $A \subset X$ is said to be compact if it is compact as a subspace.

It is easy to see that there exist topological spaces which are not compact. For instance, consider the space $(\mathbb{N}, \mathfrak{P}(\mathbb{N}))$. Clearly every subset $A \subset \mathbb{N}$ is open as a countable union of open sets and thus

$$\mathbb{N} \subset \bigcup_{n=1}^{\infty} \{n\} \tag{2.1}$$

is an open cover of $\mathbb{N}$. But it is not possible to choose a finite collection of sets from the cover in (2.1) and hence $(\mathbb{N}, \mathfrak{P}(\mathbb{N}))$ is not compact.

Definition 2.3. Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ be topological spaces and $\iota : X_1 \to X_2$ a map. We call the triple $(X_2, \mathcal{T}_2, \iota)$ a compactification of $(X_1, \mathcal{T}_1)$ if the following holds

(a) $(X_2, \mathcal{T}_2)$ is compact

(b) ι is an embedding, i.e., a homeomorphism onto its image

(c) $\overline{\iota(X_1)} = X_2$, i.e., $\iota(X_1)$ is dense in X_2

If $(X_2, \mathcal{T}_2)$ is an addition a Hausdorff space, we call $(X_2, \mathcal{T}_2, \iota)$ a T_2-compactification. Further we say that two compactifications $(X_2, \mathcal{T}_2, \iota_2)$ and $(X_3, \mathcal{T}_3, \iota_3)$ of $(X_1, \mathcal{T}_1)$ are isomorphic, if there exists a homeomorphism $\phi : X_2 \to X_3$ such that $\phi \circ \iota_2 = \iota_3$.

If (X, d) is a metric space and $U \subset X$ a subset, we can turn U into a topological space by equip U with the induced metric $\tilde{d} := d\,|_{U \times U}$. The open sets of $(U, \tilde{d})$ are then intersections of open sets of X with U. In the same spirit we can proceed if we exchange (X, d) by a topological space $(X, \mathcal{T})$. Is $U \subset X$, one can easily show that one defines via $\mathcal{T}_U := \{A \cap U \mid A \in \mathcal{T}\}$ a topology on U. This topology is called the subspace or induced topology and $(U, \mathcal{T}_U)$ is called a subspace of X.

Theorem 2.4. A subspace of a T_i-space is again a T_i space if $i \in \{2, 3\frac{1}{2}\}$.

Proof. Let X be a T_2-space and $Y \subset X$ a subspace of X. Since X is a T_2-space we can find for $x, y \in Y$ with $x \neq y$ neighbourhoods $N_x, N_y \subset X$ such that $N_x \cap N_y = \emptyset$. But $N_x \cap Y$ and $N_y \cap Y$ are also disjoint neighbourhoods of x, y in Y and thus, Y is a T_2-space. Let now be X a $T_{3\frac{1}{2}}$-space, $Y \subset X$ and $A \subset Y$ a closed set in Y. Then we can find $B \subset X$ closed such that $A = Y \cap B$. In particular, we also have $y \in Y \setminus A$ implies $y \in X \setminus B$. Since X is a $T_{3\frac{1}{2}}$-space and B closed, we can find $\varphi : X \to \mathbb{R}$ continuous, such that $\varphi|_B = 0$ and $\varphi(y) = 1$. But $\varphi|_Y$ is also a continuous function with respect to the subspace topology on Y. Hence we have $\varphi : Y \to \mathbb{R}$ continuous with $\varphi|_A = 0$ and $\varphi(y) = 1$. □

Lemma 2.5. Let $(X, \mathcal{T})$ be a T_2-space. If $K \subset X$ is a compact subspace of X, then K is closed in X.

Proof. Let X be a T_2-space and $K \subset X$ a compact subspace. It is sufficient to prove that $X \setminus K$ is open in X. For $x \in X \setminus K$ fixed and $y \in K$ we can find disjoint open neighbourhoods U_y of x and V_y of y. Note that the neighbourhood U_y of x may depend on the chosen

y and carries therefore the index y. The family $(V_y)_{y\in K}$ of such neighbourhoods is then a cover for K. Since K is assumed to be compact, there exists a finite subcover $(V_y)_{y\in K_0}$ with $K_0 \subset K$ and $|K_0| < \infty$. Consequently, the set $\cap_{y\in K_0} U_y$ is an open neighbourhood of x which is disjoint with K. But such an open neighbourhood exists for any $x \in X \setminus K$ and therefore $X \setminus K$ is open in X. □

Lemma 2.6. Let $(X_1, \mathcal{T}_1)$ be a compact topological space, $(X_2, \mathcal{T}_2)$ a topological space and $f : X_1 \to X_2$ a continuous map. Then $f(X_1)$ is a compact subspace of $(X_2, \mathcal{T}_2)$.

Proof. We have to prove that $f(X_1)$ is a compact subspace of X_2. To do so, let $(U_k)_{k\in K}$ be an open cover of $f(X_1)$ with $K \neq \emptyset$ index set. By definition of the subspace topology there are open sets V_k of X_2 such that $U_k = X_2 \cap V_k$ for all $k \in K$. The pre-images $(f^{-1}(V_k))_{k\in K}$ provide then an open cover for X_1. Since X_1 is compact by assumption, there exists $K_0 \subset K$ with $|K_0| < \infty$ such that $(f^{-1}(V_k))_{k\in K_0}$ is a finite subcover for X_1. It remains to show that $(U_k)_{k\in K_0}$ is an open cover for $f(X_1)$. Suppose $y \in f(X_1)$. Then there exists $x \in X_1$ such that $y = f(x)$ and by the previous discussion there is $k \in K_0$ such that $x \in f^{-1}(V_k)$. But then $y \in f(f^{-1}(V_k)) = V_k \cap f(X_1) = U_k$ for this $k \in K_0$. □

Lemma 2.7. Let $(X, \mathcal{T})$ be a compact T_2-space.

(a) $(X, \mathcal{T}, \mathrm{id}_X)$ is a T_2-compactification of $(X, \mathcal{T})$

(b) Each pair of T_2-compactifications of $(X, \mathcal{T})$ are isomorphic

Proof. Since id_X is a homeomorphism, the properties (a), (b) and (c) of Definition (2.3) are trivially fulfilled and hence (a) follows. To show (b) let $(X_2, \mathcal{T}_2, \iota_2)$ be an arbitrary T_2 compactification of $(X, \mathcal{T})$. Since ι_2 is continuous, we can conclude from Lemma 2.6 that the set $\iota_2(X)$ is as the image of a compact set compact and by virtue of Lemma 2.5 closed in $(X_2, \mathcal{T}_2)$. Hence we can conclude from $\iota_2(K) = \overline{\iota_2(K)} = X_2$ that ι_2 is a homeomorphism. Since $(X_2, \mathcal{T}_2, \iota_2)$ was arbitrary, any T_2-compactification of $(X, \mathcal{T})$ is isomorphic to $(X, \mathcal{T}, \mathrm{id}_X)$. □

Theorem 2.8. Let $(X, \mathcal{T})$ be a topological space. The following statements are equivalent:

(a) $(X, \mathcal{T})$ is a $T_{3\frac{1}{2}}$-space

(b) $(X, \mathcal{T})$ admits a T_2-compactification

(c) There exists an index set I and an embedding $\iota : X \to [0, 1]^I$ where $[0, 1]$ is equipped with the euclidean topology and $[0, 1]^I$ with the corresponding product topology.

Proof. We first prove (c) ⇒ (b). By virtue of the Tychonoff theorem [2] we can conclude that $[0, 1]^I$ is a compact topological space, since $[0, 1]$ is compact if equipped with the euclidean topology. Further, $[0, 1]^I$ is as a product of T_2-spaces itself a T_2-space. Hence $(\overline{\iota(X)}, \mathcal{T}^I, \iota)$ is a T_2-compactification of $(X, \mathcal{T})$. (a) ⇒ (c). Let $\mathfrak{A} := C(X, [0, 1])$ be the set of all continuous functions $\varphi : X \to [0, 1]$. Consider the map

$$\iota : X \to [0, 1]^{\mathfrak{A}} \ , \ x \mapsto \iota(x) := (\varphi(x))_{\varphi\in\mathfrak{A}} \tag{2.2}$$

where we equip $[0, 1]^{\mathfrak{A}}$ with the product topology and $\overline{\iota(X)} \subset [0, 1]^{\mathfrak{A}}$ with the subspace topology. We will now show, that ι is an embedding. For this let π_φ be the projection of

[2] Let $(X_\lambda)_{\lambda\in\Lambda}$ be an arbitrary family of compact topological spaces. Then the product space $\Pi_{\lambda\in\Lambda} X_\lambda$ is also a compact topological space. See also [33].

$[0,1]^{\mathfrak{A}}$ onto the component φ. Clearly, this implies $\pi_\varphi \circ \iota = \varphi$. Therefore we can conclude from the universal property [3] of the product topology, that ι is continuous. Suppose that there exist $x, y \in X$ with $x \neq y$ such that $\iota(x) = \iota(y)$. By definition of ι that means that $\varphi(x) = \varphi(y)$ for all $x \in \mathfrak{A}$. By assumption $(X, \mathcal{T})$ is a Tychonoff space, hence there exists a continuous function $\mathfrak{A} \ni \kappa : X \to [0,1]$ such that $\kappa(x) = 0 \neq 1 = \kappa(y)$. This is a contradiction to the assumption $\iota(x) = \iota(y)$ and thus ι is injective. Let $A \in \mathcal{T}$ and $y \in \iota(A)$. Further let $x \in A$ such that $\iota(x) = y$. Then we can separate x from the closed set $X \setminus A$ by a function $\varphi \in \mathfrak{A}$, i.e., there exists $\varphi \in \mathfrak{A}$ such that $\varphi(x) = 0$ and $\varphi(X \setminus A) = \{1\}$. Therefore $y \in \pi_\varphi^{-1}([0,1)])$ and $\pi_\varphi^{-1}([0,1)])$ is open in the product topology. In addition $y \in \pi_\varphi^{-1}([0,1)) \cap \iota(X)$ and $\pi_\varphi^{-1}([0,1)) \cap \iota(X)$ is open in the subspace topology. By construction we have

$$\iota^{-1}(\pi_\varphi^{-1}([0,1))) \cap \iota(X) \subset A \Rightarrow y \in \pi_\varphi^{-1}([0,1)) \cap \iota(X) \subset \iota(A) \tag{2.3}$$

Hence $\iota(A)$ is a neighbourhood of y and since $y \in \iota(A)$ arbitrary we have $\iota(A)$ open in the subspace topology and ι is an embedding. (b) $\Rightarrow$ (a): Let $(X, \mathcal{T})$ be a topological space with T_2-compactification $(X_2, \mathcal{T}_2, \iota_2)$. Thus $(X_2, \mathcal{T}_2, \iota_2)$ is T_2 and compact and therefore also normal. As a subspace of a normal space is not necessary normal but $T_{3\frac{1}{2}}$ we can conclude that $\iota_2(X)$ is Tychonoff. □

Lemma 2.9. Let $(X, \mathcal{T})$ be a compact topological space.

(a) If $A \subset X$ then $\overline{A}$ is compact

(b) If $(X, \mathcal{T})$ is a T_2-space, then $(X, \mathcal{T})$ is also a T_4-space. In particular, a compact topological T_2-space is also normal.

Proof. To (a): Let $(O_i)_{i \in I}$ be an open cover of $\overline{A}$. To this cover we add the open set $\overline{A}^c$ and obtain an open cover for X. Since X is a compact space there exists a finite subset of $\{O_i \in \mathcal{T} \,|\, i \in I\} \cup \{\overline{K}^c\}$ which covers X. Since $\overline{K} \cap \overline{K}^c = \emptyset$, also $\overline{K}$ has a finite cover. To (b): Obviously it is sufficient to show that $(X, \mathcal{T})$ is T_4 to obtain that X is a normal space. For this we have to show that for arbitrary $A, B \subset X$ with $\overline{A} \cap \overline{B} = \emptyset$ we can find $O_A, O_B \in \mathcal{T}$ with $\overline{A} \subset O_A$ and $\overline{B} \subset O_B$ such that $O_A \cap O_B = \emptyset$. Let $a \in \overline{A}$ be arbitrary but fixed. Since X is T_2, we can find $a \in O_a \subset \mathcal{T}$, such that for each $b \in \overline{B}$ there exists $b \in O_b^{(a)} \in \mathcal{T}$ such that $O_a \cap O_b^{(a)} = \emptyset$. Further

$$\overline{B} \subset \bigcup_{b \in \overline{B}} O_b^{(a)} \tag{2.4}$$

is an open cover of $\overline{B}$ and by the compactness of $\overline{B}$ there is finite number of points $b_j \in \overline{B}$ with $j = 1, ..., m$ such that $U_a^B = \overline{B} \subset \cup_{j=1}^m O_j^{(a)}$. Thus U_B^a is an open set which separates $a \in \overline{A}$ and $\overline{B}$. The same procedure applies for a given point $b \in \overline{B}$ and one obtains an open covering of $\overline{A}$ i.e.,

$$\overline{A} \subset \bigcup_{a \in \overline{A}} O_a^{(b)} \Longrightarrow \overline{A} \subset \bigcup_{j=1}^{n} O_{a_j}^{(b)} =: U_b^A \tag{2.5}$$

[3]Roughly speaking in the language of category theory, a universal property is a property which an object can have, that can be described solely through the morphism into the object or out of the object. For instance, the subspace topology can be regarded as a universal property, by utilising the set of all continuous functions φ that map to $(X, \mathcal{T})$. For a comprehensive treatise of universal properties we refer to [42, 43].

by the compactness of $\overline{A}$. Then we define the neighbourhood $V^A := \cap_{k=1}^{m} U_{b_k}^A$. Thus V^A is an open set which contains $\overline{A}$ and for any b_j with $j = 1, ..., m$ there is $b_j \in O_{b_j} \in \mathcal{T}$ such that $V^A \cap O_{b_j} = \emptyset$. But $\cup_{j=1}^{m} O_{b_j}$ is an open set which contains $\overline{B}$. □

Definition 2.10. Let $(X, \mathcal{T})$ be a completely regular space. We call a T_2-compactification $(\beta(X), \mathcal{T}_\beta, \iota_{\beta(X)})$ of $(X, \mathcal{T})$ a Stone-Čech compactification of $(X, \mathcal{T})$ if for all $f \in C^b(X)$ there exists a $g \in C(\beta(X), K_f)$ such that $g \circ \iota_{\beta(X)} = f$. In short: For any bounded continuous function $f : X \to \mathbb{R}$ there exists a continuous function $g : \beta(X) \to \mathbb{R}$ such that g is an extension of f to $\beta(X)$.

Corollary 2.11. Let $(X, \mathcal{T})$ be a completely regular space. The T_2- compactification $(X_1, \mathcal{T}_1, \iota)$ constructed in Theorem 2.8 is a Stone-Čech compactification.

Proof. Let $f \in C^b(X)$. Define $\alpha := \max_{x \in X} |f(x)|$ and

$$h := \frac{f(x) + \alpha}{2\alpha} \in \mathfrak{A} \tag{2.6}$$

Then $g := 2\alpha\pi_h|_{X_1} - \alpha \in C(Y)$ and fulfills $g \circ \iota = f$. □

Theorem 2.12. Let $(X, \mathcal{T})$ be a $T_{3\frac{1}{2}}$-space. Then each pair of Stone-Čech compactifications of $(X, \mathcal{T})$ are isomorphic.

Proof. Let $(X_1, \mathcal{T}_1, \iota_1)$ and $(X_2, \mathcal{T}_2, \iota_2)$ be two Stone-Čech compactifications of $(X, \mathcal{T})$, where $(X_1, \mathcal{T}_1, \iota_1)$ is as constructed as in Theorem 2.8. We have to prove the existence of a homeomorphism $\tau : X_1 \to X_2$, such that $\tau \circ \iota_1 = \iota_2$. Similar to the proof of Theorem 2.8 we denote with $\mathfrak{A}$ the set of all continuous functions from X into $[0, 1]$. By definition of a Stone-Čech compactification, we can find to $\varphi \in \mathfrak{A}$ a function $\tilde{\varphi} \in C(X_2, [0, 1])$ such that $\tilde{\varphi} \circ \iota_2 = \varphi$. Similar to the construction of ι, consider the map

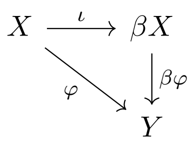
$$\tau : X_2 \to \mathfrak{A} \text{ with } x \mapsto \tau(x) := (\kappa(x)) \tag{2.7}$$

Since X_2 is T_4 it is also $T_{3\frac{1}{2}}$ and we can prove the injectivity of τ in the same way we did for ι in Theorem 2.8. The continuity of τ follows from $\pi_\varphi \circ \tau = \tilde{\varphi}$. By construction we have $\tau \circ \iota_2 = \iota_1$ what implies $\iota(X) \subset \tau(X_2)$. On the other hand we have

$$\tau(X_2) = \tau(\overline{\iota_2(X)}) \subset \overline{\tau(\iota_2(X))} = \overline{\varphi(X)}_{\varphi \in \mathfrak{A}} = \overline{\iota(X)} \tag{2.8}$$

and this $\overline{\tau(X_2)} = \overline{\iota(X)}$. Since $\tau(X_2)$ is compact in the subspace topology and $\mathfrak{A}$ is T_2, we can conclude by virtue of Lemma 2.5 that $\tau(X_2)$ is also closed, hence $\tau(X_2) = \overline{\iota(X)} = X_1$. We have thus proven that τ is a continuous bijective map from X_1 to X_2. Further we know that the inverse function of a continuous, bijective function is also continuous, therefore τ is a homeomorphism. □

The content of Theorem 2.12 and Corollary 2.11 can rephrased in terms of a univeral property of the Stone-Čech compactification. The Stone–Čech compactification of the topological space X is a compact Hausdorff space βX together with a continuous map $\iota : X \to \beta X$ that has the following universal property: any continuous map $\varphi : X \to Y$ where Y is a compact Hausdorff space, extends uniquely to a continuous map $\beta\varphi : \beta X \to Y$, i.e., $\beta\varphi \circ \iota = \varphi$. In terms of a commutative diagram this means



2.1.1 Filter convergence

Definition 2.13. Let X be a set. A system $\mathcal{F}$ of subsets of X is called a filter on X if

(1) $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$

(2) $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$

(3) $F \in \mathcal{F}$ and $F \subset G \Rightarrow G \in \mathcal{F}$

A subset $\mathcal{F}_0 \subset \mathcal{F}$ is called a filter basis or prefilter, if any element from $\mathcal{F}$ contains an element from $\mathcal{F}_0$ i.e., for every $F \in \mathcal{F}$ there exists $G \in \mathcal{F}_0$ such that $G \subset F$. A filter is called free, if $\cap_{F\in\mathcal{F}} = \emptyset$, otherwise fixed. If two filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on X are given we call $\mathcal{F}_1$ finer than $\mathcal{F}_2$ if $\mathcal{F}_2 \subset \mathcal{F}_1$. We say that $\mathcal{F}$ is an ultrafilter, if it is a maximal filter with respect to inclusion partial order, i.e., there exists no filter on X which is finer than $\mathcal{F}$.

Let $\mathcal{B}$ be a nonempty system of nonempty sets of X. Then $\mathcal{B}$ is a filter basis for some filter $\mathcal{F}$ on X, if for $B_1, B_2 \in \mathcal{B}$ there is $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$.

Example 2.14. Let X be a nonempty set.

(1) If $\emptyset \neq A \subset X$, then the system of sets

$$\mathcal{F} := \{F \subset X \mid A \subset F\} \tag{2.9}$$

is a filter on X. Obviously, the filter is fixed with $\cap_{F\in\mathcal{F}} F = A$ and $\mathcal{B} = \{A\}$ is a filter basis for $\mathcal{F}$.

(2) If X is in addition a topological space, i.e., $X = (X, \mathcal{T})$, the set $\mathcal{U}_x$ of all neighbourhoods of a point $x \in X$ is a fixed filter on X. We call this filter the neighbourhood filter of x.

(3) Let $(x_k)_{k\geq 1} \subset X$ be a sequence in X and consider the system $\mathcal{B}$ of sets defined via $B_k := \{x_i \mid k \leq i\}$ for $k \geq 1$. The system $\mathcal{B}$ does not constitute a filter but a filter basis for some filter $\mathcal{F}$ on X. We call $\mathcal{F}$ the filter induced by the sequence $(x_k)_{k\geq}$.

Definition 2.15. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{F}$ a filter on X.

(1) The filter $\mathcal{F}$ converges to $x \in X$, if $\mathcal{U}_x \subset \mathcal{F}$. In this case we call x the limit of $\mathcal{F}$ and write $\mathcal{F} \to x$.

(2) A point $x \in X$ is called an adherent point of $\mathcal{F}$, if $F \cap U$ for all $U \in \mathcal{U}_x$ and all $F \in \mathcal{F}$.

Theorem 2.16. Let $(X, \mathcal{T})$ be a topological space.

(a) Every filter $\mathcal{F}$ is contained in some ultrafilter

(b) $\mathcal{F}$ is an ultrafilter on X if and only if for each $A \subset X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Proof. We first show (a). Let denote by Γ the set of all filters on X that are finer than $\mathcal{F}$. Clearly $(\Gamma, \subset)$ is a partially ordered set. Is Γ_1 a totally ordered subset of Γ also

$$\bigcup_{\mathcal{F}\in\Gamma_1} \mathcal{F} \tag{2.10}$$

is a filter and an upper bound of Γ_1. Thus Γ contains upper bounds for every chain and therefore admits by Zorn's lemma a maximal element $\mathcal{G}$. Clearly, this element $\mathcal{G}$ is an ultrafilter. We now prove (b). Since $A \cap (X \setminus A) = \emptyset$ there cannot exist two sets $\in \mathcal{F}$ □

Lemma 2.17. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. If $\mathcal{F}$ is an ultrafilter on $\mathcal{F}$ then $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Proof. Suppose there exists $F \in \mathcal{F}$ such that $A \cap F = \emptyset$. Then $F \subset X \setminus A$ and by definition of a filter we have $X \setminus A \in \mathcal{F}$. If there is no such $F \in \mathcal{F}$, i.e., all members of the filter intersect A, the system of sets

$$\mathcal{G} := \{F \cap A \,|\, F \in \mathcal{F}\} \tag{2.11}$$

is a filter containing A and all sets from $\mathcal{F}$. Hence $\mathcal{G} = \mathcal{F}$ and $A \in \mathcal{F}$. □

The notion of compactness of a space $(X, \mathcal{T})$ and the concept of a filter are closely connected. While in metric spaces one has for $A \subset X$ that $x \in \overline{A}$ if and only if there exist some $(x_n)_{n \geq 1} \subset A$ such that $x_n \xrightarrow{n \to \infty} x \in A$, this can be replaced by an equivalent statement using ultrafilters.

Theorem 2.18. The following claims are equivalent:

(a) X is compact

(b) Any ultrafilter on X converges

Proof. Suppose that there exists an ultrafilter $\mathcal{F}$ on X that is not convergent. In this case we can find for each $x \in X$ an open neighbourhood $U(x)$ such that $U(x) \notin \mathcal{F}$. In particular, the family $(U(x))_{x \in X}$ is an open cover for X. Since X is assumed to be compact, there exist finitely many $x_1, ..., x_n \in X$, such that $U(x_1), ..., U(x_n)$ is a cover for X. From Lemma 2.17 we can conclude from $U(x_k) \notin \mathcal{F}$ that $X \setminus U(x_k) \in \mathcal{F}$. But from this we obtain

$$\mathcal{F} \ni \bigcap_{k=1}^{n} (X \setminus U_{x_k}) = X \setminus \bigcup_{k=1}^{n} U_{x_k} = X \setminus X = \emptyset \notin \mathcal{F} \tag{2.12}$$

what is a contradiction. Assume that there exists a cover $(U_i \,|\, i \in I)$ of X such that no finite subcover exists. The system of sets

$$\{X \setminus \bigcup_{i \in E} U_i \,|\, E \subset I \text{ with } |E| < \infty\} \tag{2.13}$$

constitutes a filter. Let $\mathcal{F}$ be an ultrafilter with limit x which contains this filter. But then there exists i_0 such that $x \in U_{i_0}$ and thus $U_{i_0} \in \mathcal{F}$ yielding a contradiction, since a filter can not contain a set and its complement as their intersection is empty. □

Corollary 2.19. Let $(X, \mathcal{T}_1)$ be a compact space and $f : X \to Y$ a continuous mapping. Then also $f(X)$ is compact.

Proof. Let $(U_i)_{i \in I} \subset \mathcal{T}_1$ be an open cover of $f(X)$. Since f is continuous, we have that $(f^{-1}(U_i))_{i \in I}$ is an open cover of X which contains by virtue of the compactness of X a finite subset $L \subset I$ such that also $(f^{-1}(U_i))_{i \in L}$ is an open cover of X. Hence $(U_i)_{i \in L}$ is a finite open cover of $f(X)$ and thus $f(X)$ is compact. □

Since we are mostly interested in filters and ultrafilters on $\mathbb{N}$, we will investigate the structure of these objects in more detail. In fact, filters over $\mathbb{N}$ are a special case of filters over a at most countable set [44]. The following definitions were first introduced in Choquet [20, 21], who made important contributions on the fields of ultrafilters over $\mathbb{N}$.

Definition 2.20. Let $\mathcal{F}$ be an filter over $\mathbb{N}$. We call $\mathcal{F}$

(1) δ-stable, if it is an ultrafilter and for any given countable collection of sets in $\mathcal{U}$ there exists $U \in \mathcal{U}$ which is almost contained in every set of the collection.

(2) rare, if for a given arbitrary partition $N_1, N_2, ...$ of $\mathbb{N}$ with $|N_j| < \infty$ for all i, there exists $U \in \mathcal{U}$ such that $|N_j \cap U| \leq 1$ for all j.

(3) absolute, if it is δ-stable and rare.

Even if the definition of an δ-stable ultrafilter and a rare ultrafilter seems to be quite different, it turns out that they can be placed on the same footing.

Lemma 2.21. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$.

(1) $\mathcal{U}$ is δ-stable

(2) For any partition $N_1, N_2, ...$ of $\mathbb{N}$ there exists $X \in \mathcal{U}$ such that either there exists $j \in \{1, 2, ...\}$ such that $N_j \in \mathcal{U}$ or there exists $U \in \mathcal{U}$ such that $|N_j \cap U| < \infty$ for all $j \in \{1, 2, ...\}$.

(3) For any function $f : \mathbb{N} \to [0, 1]$ there exists $X \in \mathcal{U}$ such that $\overline{f(X)}$ contains at most one accumulation point.

Lemma 2.22. Let $\mathcal{F}$ be a filter on $\mathbb{N}$. The following statements are equivalent.

(1) $\mathcal{F}$ is rare

(2) For any partition $N_1, N_2, ...$ of $\mathbb{N}$ into finite intervals i.e., for all $k \geq 1$ there exist $n_1^{(k)}, n_2^{(k)} \in \mathbb{N}$ with $n_1^{(k)} < n_2^{(k)}$ and $N_k = \mathbb{N} \cap [n_1^{(k)}, n_2^{(k)}]$, such that there exists $X \in \mathcal{F}$ with $|N_k \cap U| \leq 1$ for all k.

Concerning the existence of ultrafilters that share the properties of δ-stability, rareness and absolutness, Chouqet proved [21] modulo the continuum hypothesis [4], that all possibilities can in fact occur. In particular, there exist absolute ultrafilters, ultrafilters that are δ-stable and not rare as well as ultrafilters that are neither stable nor rare.

2.1.2 Net convergence

As we have already seen, one possibility to introduce a notion of convergence is via filters and ultrafilters. Another possibility is via nets and subsets which is more popular in the context of C^*-algebras.

Definition 2.23. A binary relation in a set X is given by a subset $R \subset X \times X$. We will write $x \leq y$ to indicate that $(x, y) \in R$. An order in X is a binary relation, denoted by $\leq$ which is transitive, reflexive and antisymmetric. In this case, we say that $(X, \leq)$ is an ordered set. Given a subset $Y \subset X$, we call $x \in X$ a majorant for Y if $y \leq x$ for every $y \in Y$. We say that an ordering is filtering upward, if every pair in X has a majorant.

[4] The continuums hypothesis is a hypothesis in set theory and deals with the possible sizes of infinite sets. In particular, it states that there is no set whose cardinality is strictly between that of the integers and the real numbers.

Definition 2.24. A net in a space X is a pair (Λ, i), where Λ is a non-empty upward-filtering ordered set and $i : \Lambda \to X$ is a map. A net is denoted by $(x_\lambda)_{\lambda\in\Lambda}$ where we put $x_\lambda = i(\lambda)$ and indicate the domain of i. Is in addition $(X, \mathcal{T})$ a topological space, we say that a net $(x_\lambda)_{\lambda\in\Lambda}$ converges to $x \in X$, written $(x_\lambda)_{\lambda\in\Lambda} \to x$ if for any neighbourhood U of x there exists a $i_0 \in \Lambda$ such that $x_i \in U$ for $i \geq i_0$.

Clearly the most important example arises if we set $\Lambda = \mathbb{N}$ what yields the common concept of a sequence in X. Sequences are sufficient to handle all convergence problems in spaces which satisfy the first axiom of countability which includes all metric spaces. Apart from that, there are spaces which require the more general notion of nets. These includes for instance Hilbert spaces in the weak topology, which arising frequently in the context of C^*- and von Neumann algebras. That the notion of a net is indeed the correct instrument in order to characterize convergence is the content of the following

Theorem 2.25. Let X, Y be topological spaces.

(a) $x \in \overline{A}$ for $A \subset X$ if and only if there exists a net $(x_\lambda)_{\lambda\in\Lambda}$ with $x_\lambda \in A$ such that $(x_\lambda)_{\lambda\in\Lambda} \to x$.

(b) A function $f : X \to Y$ is continuous in $x \in X$ if and only if for any net $(x_\lambda)_{\lambda\in\Lambda} \subset X$ with $(x_\lambda)_{\lambda\in\Lambda} \to x$ we have $(f(x_\lambda))_{\lambda\in\Lambda} \to f(x)$ in Y.

So far, we have introduced two different concepts to define a notion of convergence. Even not further used in this thesis, we will show for the sake of completeness that they are equivalent in a very strong way. To do so, we will first describe a procedure for producing a filter from a given net and as well as for producing nets from a given filter. We will then show that the convergence properties of one carry over to the other and vice versa. If X is a set and $\mathcal{F}$ a filter on X, we define an ordering $\leq$ on $\mathcal{F}$ by $F_1 \leq F_2$ if $F_1 \supset F_2$ for $F_1, F_2 \in \mathcal{F}$. This turns $(\mathcal{F}, \leq)$ in a directed set. Any net $i : \mathcal{F} \to X$ with the property $i(F) \in F$ for every $F \in \mathcal{F}$ is called a derived net of $\mathcal{F}$. It is important to note that derived nets are not unique. Further, if $A \subset X$ and $i : \Lambda \to X$ a net, we say that i is eventually in A, if A contains some tail of the net, i.e., there exists $d \in \Lambda$ such that $T_d := \{i(e) \mid d \leq \lambda \in \Lambda\} \subset A$. If Λ is a directed set and $i : \Lambda \to X$ a net on X, we define

$$\mathcal{F}_i := \{F \subset X \mid i \text{ is eventually in} F\} \tag{2.14}$$

and call $\mathcal{F}_{(\Lambda,i)}$ a derived filter of the net (Λ, i). It can be easily seen that the derived filter $\mathcal{F}_{(\Lambda,i)}$ is indeed a filter in the sense of Definition 2.13. We have $\emptyset \notin \mathcal{F}_{(\Lambda,i)}$ and it is closed upwards, i.e., if (Λ, i) is eventually in F and $F \subset G$, then (Λ, i) is eventually in G. It remains to show that $\mathcal{F}_{(\Lambda,i)}$ is closed under finite intersections. For $F_1, F_2 \in \mathcal{F}_{(\Lambda,i)}$, we know that there exist $d_1, d_2 \in \Lambda$ such that T_{d_1} and T_{d_2} are tails of (Λ, i) in F_1 and F_2 respectively. Since Λ is a directed set, we can find $d \in \Lambda$ such that $d_1 \leq d$ and $d_2 \leq d$. Then we have $T_d \subset F_1 \cap F_2$ and thus $F_1 \cap F_2 \in \mathcal{F}_{(\Lambda,i)}$.

Theorem 2.26. Let $(X, \mathcal{T})$ be a topological space and $x \in X$.

(1) If (Λ, i) is a net in X, then $(x_\lambda)_{\lambda\in\Lambda} \to x$ if and only if the derived filter $\mathcal{F}_{(\Lambda,i)} \to x$

(2) If $\mathcal{F}$ is a filter on X, then $\mathcal{F} \to x$ if and only if every derived net of $\mathcal{F}$ converges to x.

Proof. To (1): This follows directly from the definition. To (2): Assume that $\mathcal{F} \to x$ and let (Λ, i) be a derived net of $\mathcal{F}$. Further let $U \in \mathcal{T}$ such that $x \in U$. We have to show that there exists a tail of the net which is contained in U. Since $\mathcal{F} \to x$ implies that $\mathcal{F}$ contains every open set containing x, we have $U \in \mathcal{F}$. We will now prove that $T_U \subset U$. For $V \in \mathcal{F}$ with $U \leq V \Leftrightarrow V \subset U$, we have $i(V) \in V \subset U$. Hence $T_U \subset U$ and $(x_\lambda)_{(\Lambda,i)} \to x$. To show the other implication, assume that every derived net (Λ, i) in X converges to $x \in X$. Suppose that $\mathcal{F}$ not converges to x. Then we can find $A \subset X$ containing an open set U with $x \in U$ such that $A \notin \mathcal{F}$. More precise, $U \notin \mathcal{F}$ or equivalently $F \neq U$ for all $F \in \mathcal{F}$. Let i be any derived net of $\mathcal{F}$ such that $i(V) \in V \setminus U$ for all $V \in \mathcal{F}$. Then, this derived net does not converge to x, since no point of $\mathcal{F}$ gets mapped into U by i. This contradicts the assumption, so that all derived nets of $\mathcal{F}$ do converge to x. □

2.1.3 Seminorms and local convexity

Often when vector spaces appear in the context of analysis, the notion of convergence of a sequence is defined with respect to a given norm. For instance, a popular example is the uniform convergence in $C[0,1]$ by virtue of the supremum norm $|| \cdot ||_\infty$. However, within this framework the concept of pointwise convergence of a sequence of functions $(f_k)_{k\geq 1} \subset C[0,1]$ can not be formulated. But in order to be able to talk about convergence, a norm is far too much, i.e., a topological structure is already sufficient. Therefore, for $t \in [0,1]$ and $(f_k)_{k\geq 1} \subset C[0,1]$ define the map $p_t(f_k) := |f_k(t)|$, that is, the evaluation of the function f_k at point t. The statement that the sequence of functions $(f)_{k\geq 1}$ converges pointwise to $f \equiv 0$, can then be stated as $p_t(f_k) \xrightarrow{k\to\infty} 0$ for all $t \in [0,1]$. In the following we mean by a vector space a vector space over $\mathbb{K}$, where $\mathbb{K} = \{\mathbb{R}, \mathbb{C}\}$.

Definition 2.27. Let X be a vector space. A seminorm $p : X \to \mathbb{R}$ is a function on X such that

(a) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$

(b) $p(\lambda x) = |\lambda|\, p(x)$ for all $\lambda \in \mathbb{K}$ and $x \in X$.

A family $\mathcal{P}$ of seminorms on X is called separating if for each $x \neq 0$ there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$. A convex set $A \subset X$ is called absorbing for X, if every $x \in X$ lies in tX for some $t = t(x) > 0$. Further we call A balanced, if $\lambda A \subset A$ for any $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$. The Minkowski functional μ_A of A is defined for $x \in X$ via

$$\mu_A(x) := \inf\{t > 0 \,|\, t^{-1}x \in A\} \tag{2.15}$$

Lemma 2.28. Suppose that p is a seminorm on a vector space X. Then

(a) $|p(x) - p(y)| \leq p(x-y)$

(b) $\{x \in X \,|\, p(x) = 0\}$ is a linear subspace of X

(c) The set $B := \{x \in X \,|\, p(x) < 1\}$ is convex, balanced,absorbing and $p = \mu_B$

Proof. To (a). By the subadditivity of p we have

$$p(x) = p(x+y-y) \leq p(x-y) + p(y) \Leftrightarrow p(x) - p(y) \leq p(x-y) \tag{2.16}$$

Since this also holds if the roles of x, y are interchanged and using $p(x-y) = p(y-x)$, the claim follows. To (b). If $p(x) = p(y) = 0$ and $\lambda_1, \lambda_2 \in \mathbb{K}$, the non negativity of a seminorm implies $0 \leq p(\lambda_1 x + \lambda_2 y) \leq |\lambda_1|\, p(x) + |\lambda_2|\, p(y) = 0$. To (c). If $x \in B$, then also $\lambda x \in B$ for

$|\lambda| \leq 1$ since $p(\lambda x) = |\lambda|\, p(x) \leq p(x) < 1$. Therefore B is balanced. Further, if $x, y \in B$ and $t \in (0,1)$, we have

$$p(tx + (1-t)y) \leq tp(x) + (1-t)p(y) < 1 \tag{2.17}$$

what yields the convexity of B. If $x \in X$ and $s > p(x)$, then $p(s^{-1}x) = s^1 p(x) < 1$, hence B is absorbing and $\mu_B(x) \leq s$, implying $\mu_B \leq p$. But if $0 < t \leq p(x)$ then $p(t^{-1}x) \geq 1$ and thus $t^{-1}x \notin B$. Consequently $p(x) \leq \mu_B(x)$ what yields $p(x) = \mu_B(x)$. □

Theorem 2.29. Let X be a vector space and suppose $\mathcal{P}$ is a separating family of seminorms on X. For each $p \in \mathcal{P}$ and $n \in \mathbb{N}$ associate the set

$$U_{p,n} := \{x \in X \,|\, p(x) < \frac{1}{n}\} \tag{2.18}$$

Let $\mathcal{B}$ be the collection of all finite intersections of the sets $U_{p,n}$. Then $\mathcal{B}$ is a convex balanced local base for a topology $\mathcal{T}$ on X, which turns X into a locally convex space such that the following properties hold

(a) every $p \in \mathcal{P}$ is continuous with respect to $\mathcal{T}$

(b) $E \subset X$ is bounded if and only if every $p \in \mathcal{P}$ is bounded on E.

Proof. Let $A \subset X$. We declare the set A to be open if and only if it is an union of translates of members of $\mathcal{B}$. By construction this defined a translation-invariant topology $\mathcal{T}$ on X as each member if $\mathcal{B}$ is convex and balanced and $\mathcal{B}$ is a local base for $\mathcal{T}$. Suppose now that $x \in X \setminus \{0\}$. Since the family of seminorms is separating, there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$. If $np(x) > 1$ then $x \notin U_{p,n}$ and therefore 0 is not contained in the neighbourhood $x \setminus U_{p,n}$ of x and consequently x is not in the closure of $\{0\}$. Thus $\{0\}$ is a closed set and by the translation invariance of $\mathcal{T}$, every set of the form $\{x\}$ with $x \in X$ is a closed set. Further we have to prove that the vector space operations addition and scalar multiplication are continuous with respect to $\mathcal{T}$. Let N_0 be a neighbourhood of 0 in X. Then there exists $p_1, ..., p_n \in \mathcal{P}$ and $n_1, ..., n_n \in \mathbb{N}$ such that

$$N \supset U_{p_1,n_1} \cap \cdots \cap U_{p_n,n_n} \tag{2.19}$$

Define $U := U_{p_1,n_1} \cap \cdots \cap U_{p_n,n_n}$. Since any seminorm $p \in \mathcal{P}$ is subadditive, also $U + U \subset N$, hence addition is continuous. For $x \in X$ and $\lambda \in \mathbb{K}$ let U and N as be constructed in (2.19). Then there exists $s > 0$ such that $x \in sU >$. Set $t := s(1 + |\lambda|s)^{-1}$. If $y \in x + tU$ and $|\kappa - \lambda| < s^{-1}$ we have $\kappa y - \lambda x = \kappa(x - y) + (\kappa - \lambda)x$ which lies in

$$|\kappa| tU + |\kappa - \lambda| sU \subset U + U \subset N \tag{2.20}$$

since $|\kappa|t \leq 1$ and U is by assumption balanced. Consequently scalar multiplication is continuous. Summing up, X is a locally convex space. Hence by Lemma 2.28 (a) the seminorm p is continuous on X with respect to $\mathcal{T}$. To show (b), suppose that E is bounded and fix $p \in \mathcal{P}$. Since $U_{p,1}$ is a neighbourhood of 0, we have $E \subset kU_{p,1}$ for a sufficient large $k < \infty$. Therefore $p(x) < k$, for every $x \in E$ and every $p \in \mathcal{P}$ is bounded on E. Conversely assume that $E \subset X$ such that all $p \in \mathcal{P}$ are bounded on E. Let U be a neighbourhood of 0 such that $U \supset U_{p_1,n_1} \cap ... \cap U_{p_n,n_n}$. Then there are numbers $M_i < \infty$ such that $p_i < M_i$ on E for $i = 1, ..., n$. If $\ell > M_i n_i$ it follows $E \subset \ell U$, so that E is bounded. □

2.2 C*-algebras

Definition 2.30. A Banach algebra is a Banach space $(A, ||\cdot||)$ that is in addition an algebra in which

$$||ab|| \leq ||a||\,||b|| \quad \forall a, b \in A \tag{2.21}$$

An involution on an algebra A is a $\mathbb{R}$-linear map $* : A \to A$ with $a \mapsto a^*$ such that $a^{**} = a$, $(ab)^* = b^*a^*$ and $(\lambda a)^* = \overline{\lambda}a^*$ for all $a, b \in A$ and $\lambda \in \mathbb{C}$. An algebra with an involution is called a $*$-algebra. A C^*-algebra is a Banach algebra A with an involution in which

$$||a^*a|| = ||a||^2 \quad \forall a \in A \tag{2.22}$$

Example 2.31. There are two main examples of C^*-algebras that will appear frequently in this thesis.

(1) Let X be a locally compact Hausdorff space and denote with $C_0(X)$ the set of all continuous functions $f : X \to \mathbb{C}$ that vanish at infinity. If one equips this set with pointwise operations, i.e., $(fg)(x) = f(x)g(x)$ and $(\lambda f + g)(x) = \lambda f(x) + g(x)$ this turns $C_0(X)$ into an algebra. Furthermore, there is a natural involution inherited from $\mathbb{C}$, namely $f^*(x) = \overline{f(x)}$ and a natural norm $||f||_\infty = \sup_{x\in X}|f(x)|$. The algebra $C_0(X)$ is unital if and only if X is compact. In this case the identity element is given by the function $f(x) = 1$ for all $x \in X$. The most important property of $(C_0(X), ||\cdot||_\infty)$ is that it constitutes a commutative C^*-algebra.

(2) Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ the set of all bounded linear operators from $\mathcal{H}$ to itself, with the obvious algebraic operations $(+, \cdot)$ and the involution given by the adjoint. The norm is taken to be the operator norm, i.e., for $x \in B(\mathcal{H})$ we have $||x|| = \sup\{||x\xi|| \,|\, \xi \in \mathcal{H}\,,\, ||\xi|| = 1\}$. It is a unital C^*-algebra, where the unit is given by the identity operator $1_{\mathcal{H}}$. If $\dim(\mathcal{H}) > 1$, $B(\mathcal{H})$ is a non-commutative C^*-algebra.

To be more precise, Definition 2.30 is the definition of an abstract C^*-algebra, which is strongly motivated by the structure of $B(\mathcal{H})$, as introduced in Example 2.31. Moreover, each operator norm-closed $*$-algebra in $B(\mathcal{H})$ is a C^*-algebra. In Theorem 2.40 we will see, that this are in fact all possible examples.

Definition 2.32. Let A be an algebra and $S \subset A$ a subset. We call the set

$$S^{'} := \{a \in A \,|\, sa = as \;\forall s \in S\} \tag{2.23}$$

the commutant of S. The double commutant SS'' of S is defined by $S' = (S')'$.

Definition 2.33. A homomorphism between C^*-algebras A, B is a linear map $\varphi : A \to B$ such that for all $a_1, a_2 \in A$ we have $\varphi(a_1a_2) = \varphi(a_1)\varphi(a_2)$ and $\varphi(a^*) = \varphi(a)^*$. An isomorphism between two C^*-algebras is an invertible homomorphism. If A and B are isomorphic as C^*-algebras, we write $A \cong B$.

2.2.1 The structure of states

Definition 2.34. A state on a C^*-algebra A is bounded linear map $\phi : A \to \mathbb{C}$ that satisfies

(1) $\phi(a^*a) \geq$ for all $a \in A$

(2) $||\phi|| = 1$, i.e., it is normalized

A state $\phi : A \to \mathbb{C}$ is called normal if for each collection $\{p_i\}$ of mutually orthogonal projectors of A one has

$$\phi\left(\sum_i p_i\right) = \sum_i \phi(p_i) \tag{2.24}$$

The weak operator topology on $B(\mathcal{H})$ is defined by saying that $a_\lambda \to a$ for some net $(a_\lambda)_{\lambda\in\Lambda}$ if $\langle v, (a_\lambda - a)w\rangle \to 0$ for all $v, w \in \mathcal{H}$. The strong operator topology on $B(\mathcal{H})$ yields convergence of some net $(a_\lambda)_{\lambda\in\Lambda}$ if $||(a_\lambda - a)v|| \to 0$ for each $v \in \mathcal{H}$. This can be seen also in the context of locally convex topologies. One can equivalently define the weak operator topology on $B(\mathcal{H})$ as the locally convex topology induced by the family of seminorm $\{p_{v,w}(A)\,|\, v, w \in \mathcal{H}\}$ with $p_{v,w}(A) := |\langle Av, w\rangle|$ and the strong operator topology as the topology induced by the family of seminorms $\{p_v(A)\,|\, v \in \mathcal{H}\}$ where $p_v(A) = ||Av||$.

Theorem 2.35. The following conditions on a linear functional on $B(\mathcal{H})$ are equivalent

(1) $\phi(x) = \sum_{k=1}^{n} \langle x\xi_k | \eta_k\rangle$ for some $\xi_k, \eta_k \in H$ and all $x \in B(\mathcal{H})$

(2) ϕ is weakly continuous

(3) ϕ is strongly continuous

Proof. The implications (1) ⇒ (3) ⇒ (3) are clear since the weak topology is weaker than the strong topology which is itself weaker than the norm topology. We proceed by showing that (3) $\Rightarrow$ (1). Let ϕ be strongly continuous. Then we can find vectors $\xi_1, ..., \xi_n \in \mathcal{H}$ such that $\max_k ||x\xi_k|| \le 1$ implies $|\phi(x)| \le 1$ for all $x \in B(\mathcal{H})$. Therefore we have

$$|\varphi(x)|^2 \le \sum_{k=1}^{n} ||x\xi_k||^2 \tag{2.25}$$

Define $\mathcal{K} := \oplus_{k=1}^{n} \mathcal{H}$ to be the orthogonal sum of n copies of $\mathcal{H}$, set $\xi = \xi_1 \oplus \cdots \oplus \xi_n \in \mathcal{K}$ as the orthogonal sum of the ξ_k. Further define a functional ρ on the subspace of $\mathcal{K}$ consisting of vectors of the form $\tau(x)\xi$ for $x \in B(\mathcal{H})$ where $\tau : B(\mathcal{H}) \to B(\mathcal{H}^n)$ with $\tau(x)_{kk} = x$ and $\tau(x)_{kl} = 0$ if $k \neq l$. The functional ψ acts as $\psi(\tau(x)) = \varphi(x)$ and by (2.25) it is also continuous, since $|\psi(\tau(x)\xi)|^2 \le ||\tau(x)\xi||^2$. By virtue of the Hahn-Banach theorem we can extend ψ to the whole space $\mathcal{H}^n$. By the Riesz-representation theorem, there is a vector $\eta = (\eta_1, ..., \eta_n) \in \mathcal{H}^n$ such that $\psi = \langle \cdot \,|\, \eta\rangle$. In particular, we have

$$\varphi(x) = \psi(\tau(x)\xi) = \langle \tau(x)\xi|\eta\rangle = \sum_{k=1}^{n} \langle x\xi_k|\eta_k\rangle \tag{2.26}$$

□

Corollary 2.36. Every strongly closed, convex set in $B(\mathcal{H})$ is weakly closed. In particular, every strongly closed subspace of $B(\mathcal{H})$ is weakly closed.

Theorem 2.37 ([36]). For $\omega \in S(B(\mathcal{H}))$ the following are equivalent

(a) ω is normal

(b) $\omega(a) = \lim_\lambda \omega(a_{\lambda)}$ for all $a_\lambda \nearrow a$

(c) $\omega(a) = \mathrm{tr}[\rho a]$ for some density operator $\rho \in \mathcal{D}(\mathcal{H})$

(d) ω is σ-weakly continuous

Corollary 2.38. Let $A \subset B(\mathcal{H})$ be a unitary C^*-algebra in $B(\mathcal{H})$ ans let ω_A be a pure state on A. The set

$$S_A := \{\omega \in S(B(\mathcal{H})) \,|\, \omega_{|A} = \omega_A\} \tag{2.27}$$

is a compact convex subset of the total state space $S(B(\mathcal{H}))$. Further the extreme boundary $\partial_e S_A$ consists of pure states on $B(\mathcal{H})$, i.e., $\partial_e S_A \subset P(B(\mathcal{H}))$. In particular ω_A has a unique extension to a state on $B(\mathcal{H})$ if and only if it has a unique pure extension.

Proof. It is clear that the set S_A is w^*-compact and convex. Let $\omega \in \partial_e S_A$ and suppose ω is not pure, i.e., $\omega = t\omega_1 + (1-t)\omega_2$ for some $t \in (0,1)$ and $\omega_1, \omega_2 \in S(B(\mathcal{H}))$. By assumption we have that $\omega_{|A} = t\omega_{1|A} + (1-t)\omega_{2|A}$ is pure on A and consequently we have $\omega_A = \omega_{1|A} = \omega_{2|A}$ and $\omega_1, \omega_2 \in S_A$. Since $\omega \in \partial_e S_A$ this implies $\omega = \omega_1 = \omega_2$ and thus ω is pure on $B(\mathcal{H})$. To prove the last claim, observe that S_A only contains one element if and only if $\partial_e S_A$ contains one element. □

Theorem 2.39. There is a bijective correspondence between pure states on $\ell^\infty(\mathbb{N})$ and ultrafilters on $\mathbb{N}$. In particular, any pure state $\omega \in S(\ell^\infty(\mathbb{N}))$ is of the form

$$\omega(\mathrm{diag}(x)) = \lim_{\mathcal{U}} x \tag{2.28}$$

for all $x \in \ell^\infty(\mathbb{N})$ and some unique ultrafilter $\mathcal{U}$.

Proof. Suppose that ω is a pure state on $\ell^\infty(\mathbb{N})$. For $A \subset \mathbb{N}$ denote by Π_A the orthogonal projection of $\ell^2(\mathbb{N})$ onto $\overline{\mathrm{span}}(\{e_j \,|\, j \in A\})$. Now define $\mathcal{U} := \{A \subset \mathbb{N} \,|\, \omega(\Pi_A) = 1\}$. We are now going to show that $\mathcal{U}$ is an ultrafilter. By Theorem 2.8, the Stone-Čech compactification of $\mathbb{N}$ is a maximal compact Hausdorff space $\beta\mathbb{N}$, which contains $\mathbb{N}$ as a dense subset. By the universal property of $\beta\mathbb{N}$, the space $\ell^\infty(\mathbb{N})$ is isometric isomorphic to $C(\beta\mathbb{N})$. By virtue of the Riesz representation theorem, positive functionals on $C(\beta\mathbb{N})$ can be identified with regular Borel measures. Thus, ω corresponds to a probability measure on $\beta\mathbb{N}$. Since we assume ω to be pure, this measure has only mass in one point of $\beta\mathbb{N}$. Consequently we have $\omega(\Pi_A) = \omega(\Pi_A)^2$ which implies $\omega(P_A) \in \{0,1\}$. Similar we have $\omega(\Pi_{A\cap B}) = \omega(\Pi_A)\omega(\Pi_B)$ for any $A, B \subset \mathbb{N}$ and therefore $\mathcal{U}$ must be an ultrafilter. We now show the uniqueness. Every $x \in \ell^\infty(\mathbb{N})$ can be approximated in norm by simple functions, i.e., finite linear combinations of indicator functions 1_{A_j} for disjoint subsets $A_i \subset \mathbb{N}$. By definition we have

$$\omega(\mathrm{diag}(1_{A_i})) = \omega(\Pi_{A_i}) = \lim_{\mathcal{U}} 1_{A_i} = \begin{cases} 1 & \text{if } A_i \in \mathcal{U} \\ 0 & \text{otherwise} \end{cases} \tag{2.29}$$

Hence (2.28) holds for all indicator functions. Since functions of this type are dense [34], the claim holds for arbitrary $x \in \ell^\infty(\mathbb{N})$. It follows that two distinct pure states must correspond to distinct ultrafilters. □

2.2.2 Classification of C^*-algebras

Theorem 2.40. Each C^*-algebra A is isomorphic to a norm-closed $*$-algebra in $B(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.

The proof of Theorem 2.40 relies on the so called GNS-construction. This ingenious construction discovered independently by Gelfand and Naimark [45, 46] and I. Segal [47]is one of the most fundamental ideas of the theory of operator algebras and provides a method for manufacturing representations of C^*-algebras.

Definition 2.41. A representation of a C^*-algebra A is a $*$-homomorphism from A to $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Two representations π and ρ of A on Hilbert spaces X and Y respectively are unitarily equivalent if there is a unitary operator $U \in B(X, Y)$ with $U\pi(x)U^* = \rho(x)$ for all $x \in A$. A subrepresentation of a representation π on $\mathcal{H}$ is the restriction of π to a closed invariant subspace of $\mathcal{H}$. A representation is irreducible if it has no closed invariant subspaces. If $I \neq \emptyset$ index set and π_i a representation of A on $\mathcal{H}_i$, then the sum $\oplus_i \pi_i$ of the π_i is the diagonal sum acting on $\oplus_i \mathcal{H}_i$. If each π_i is equivalent to a fixed representation ρ, then $\oplus_i \pi_i$ is called amplification of ρ by $|I|$. A representation with kernel 0 is called faithful. We call a representation π cyclic, if its carrier space $\mathcal{H}$ contains a cyclic vector ξ for π, i.e., the closure of $\pi(A)\xi$ coincides with $\mathcal{H}$ and it is called nondegenerate that $\pi(a)v = 0$ for all $a \in A$ and $v \in \mathcal{H}$ implies $v = 0$.

Theorem 2.42 (GNS construction). Let A be a unital C^*-algebra [5] and ϕ a state on A. There exists a cyclic representation π_ϕ of A on a Hilbert space $\mathcal{H}_\phi$ with cyclic unit vector ξ_φ such that

$$\phi(a) = \langle \xi_\phi, \pi_\phi(a)\xi_\phi \rangle \ , \ a \in A \tag{2.30}$$

Proof. We first consider the special case where $\phi(a^*a) > 0$ for all $a \in A \setminus \{0\}$. Define a sesquilinear from $(\cdot, \cdot)$ on A by $(a, b) := \phi(a^*b)$. Since ϕ is a state, the sesquilinear form is positive and we can complete A in the ensuing norm given by $||a||_\phi = \sqrt{\phi(a^*a)}$ to a Hilbert space $\mathcal{H}_\phi$. For each $a \in A$ define the map $\pi_\phi(a) : A \to A$ via $\pi_\phi(a)b = ab$, i.e., the left multiplication by $a \in A$. Regarding A as a dense subspace of $\mathcal{H}_\phi$ this defines an operator $\pi_\phi(a)$ on a dense domain in $\mathcal{H}_\phi$. This operator is bounded since $||\pi_\phi(a)|| \leq ||a||$. Hence we can extend $\pi_\phi(a)$ from A to $\mathcal{H}_\phi$ by continuity and we obtain a map $\pi_\phi : A \to B(\mathcal{H}_\phi)$. A direct calculation shows that π_ϕ is indeed a representation. The cyclic vector ξ_ϕ is given by the unit $1 \in A$, seen as an element of $\mathcal{H}_\phi$. Clearly, this element is cyclic and we have

$$||\xi_\phi||^2 = \langle \xi_\phi, \xi_\phi \rangle = \phi(1^*1) = \phi(1) = 1 \text{ and } \langle \xi_\phi, \pi_\phi(a)\xi_\phi \rangle = \phi(1^*a1) = \phi(a) \tag{2.31}$$

Suppose now that there exists states ϕ on A such that $\phi(a^*a) = 0$ for some $a \in A$. For an arbitrary state ϕ define

$$N_\phi := \{a \in A \,|\, \phi(a^*a) = 0\} \tag{2.32}$$

and consider the quotient space A/N_ϕ. If $[a]_\phi$ denotes the image of $a \in A$ under the projection in A/N_ϕ we can define an inner product on A/N_ϕ via $\langle [a]_\phi, [b]_\phi \rangle := \phi(a^*b)$. This form is well defined and positive definite, hence we can define the Hilbert space $\mathcal{H}_\phi$ as the completion of A/N_ϕ with respect to this inner product. Furthermore define

$$\pi_\phi(a) : A/N_\phi \to H_\phi \text{ with } [b]_\phi \mapsto \pi_\phi(a)[b]_\phi := [ab]_\phi \tag{2.33}$$

This map $\pi_\phi(a)$ is well defined for each $a \in A$, since N_ϕ is a left ideal in A. If we define in addition $\xi_\phi = [1]_\phi$, the claim follows. □

[5] The theorem remains true even if A is not unital.

It is important to note that the proof of Theorem 2.42 also proves Theorem for the case that there exists a state $\phi \in \mathcal{S}(A)$ with $\phi(a^*a) > 0$ for all $a \in A$. In this case $\pi_\phi(a) = 0$ implies $||\pi_\phi(a)\xi_\phi||^2 = 0$, where $||\pi_\phi(a)\xi_\phi|| = \langle \xi_\phi, \pi_\phi(a^*a)\xi_\phi \rangle = \phi(a^*a)$. Thus π_ϕ is faithful. Another important property of the GNS-construction is the link between the purity of a state ϕ and irreducibility of the corresponding representation π_ϕ. To be more precise, we have the following

Theorem 2.43. Let A be a unital C^*-algebra and $\pi = (\pi, \mathcal{H})$ be a representation. The following claims are equivalent

(a) π is irreducible

(b) $\pi(A)' = \mathbb{C} \cdot 1$

(c) $\pi(A)'' = B(\mathcal{H})$

(d) Every vector in $\mathcal{H}$ is cyclic for $\pi(A)$

Furthermore, if $\phi \in \mathcal{S}(A)$ then ϕ is pure if and only if the corresponding GNS-representation π_ϕ is irreducible.

Proof. Suppose π is irreducible but $\pi(A)' \neq \mathbb{C} \cdot 1$. Then $\pi(A)'$ must contain a nontrivial self-adjoint element $a \in A$, since it is a $*$-algebra and therefore also a nontrivial projection P. But if $P \in \pi(A)'$, then $P(\mathcal{H})$ is stable under $\pi(A)$ and thus π cannot be irreducible. Thus (a) $\Rightarrow$ (b). Conversely assume that $\pi(A)' = \mathbb{C} \cdot 1$. Then π must be irreducible, since if not, any projection onto some proper stable subspace $K \subset A$ for π would be a nontrivial element of $\pi(A)'$. The equivalence (b) $\Rightarrow$ (c) is clear, since $(\mathbb{C} \cdot 1)' = B(\mathcal{H})$. Assume that there exists $v \in \mathcal{H}$ which is not cyclic for π. Then $\overline{\pi(A)v}$ would be a proper $\pi(A)$-stable subspace of $\mathcal{H}$, so that (a) $\Rightarrow$ (d). The converse is obvious, since $K \subset A$ is stable for $\pi(A)$, then (d) cannot hold. □

Corollary 2.44. Let A be a unital C^*-algebra and $\pi : A \to B(\mathcal{H})$ cyclic representation on a Hilbert space $\mathcal{H}$. If $v \in \mathcal{H}$ is a cyclic unit vector for π, then

$$\phi(a) := \langle v, \pi(a)v \rangle \tag{2.34}$$

is a state on A, whose GNS-representation π_ϕ is unitarily equivalent to π.

Proof. Define a map $u : \mathcal{H}_\phi \to \mathcal{H}$ on the dense subspace $\pi_\phi(A)\xi_\phi$ of H by $u\pi_\phi(a)\xi_\phi = \pi(a)v$, where $\mathcal{H}_\phi$ is constructed as in the proof of Theorem 2.42. From this we obtain

$$||\pi_\phi(a)\xi_\phi||^2 = \phi(a^*a) = \langle v, \pi(a^*a)v \rangle = ||\pi(a)v||^2 \tag{2.35}$$

Hence u is a well defined map and an isometry, such that we can extent it to $\mathcal{H}_\phi$ by continuity. Further $\mathrm{im}(u) = \overline{\pi(A)v}$ which coincides with $\mathcal{H}$, since v is cyclic by assumption. Consequently u is a surjective isometry, i.e., unitary. In addition we have for $a, b \in A$

$$u\pi_\phi(a)\pi_\phi(b)\xi_\phi = \pi(a)\pi(b)v = \pi(a)u\pi_\phi(b)\xi_\phi \tag{2.36}$$

Therefore $u\pi_\phi(a) = \pi(a)u$ on the dense subspace $\pi_\phi(A)\xi_\phi$ and thus they coincide everywhere. □

Corollary 2.45. Let A be a unital C^*-algebra and $\pi : A \to B(\mathcal{H})$ nondegenerate representation on a Hilbert space $\mathcal{H}$. Then π is a direct sum of cyclic representations of A.

Proof. Let $I \neq \emptyset$ be some index set and consider families of vectors $(v_i)_{i\in I} \subset \mathcal{H}$ of nonzero vectors v_i such that

$$\langle \pi(a)v_i, \pi(b)v_j \rangle = 0 \tag{2.37}$$

for all $a, b \in A$ and $i, j \in I$ with $i \neq j$. Such families are partially ordered by inclusion and by virtue of Zorn's Lemma we can conclude that there exists a maximal such family. Suppose that $(v_i)_{i\in I}$ is such a maximal family. We define $\mathcal{H}_i := \overline{\pi(A)v_i} \subset H$. Since π is a homomorphism, each $\mathcal{H}_i$ is stable under $\pi(A)$ and thus the restriction $\pi_i(a)$ of $\pi(a)$ to $\mathcal{H}_i$ defines a representation of A itself, which is cyclic by construction. Hence

$$\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i \quad \text{and} \quad \pi = \bigoplus_{i\in I} \pi_i \tag{2.38}$$

□

Proof of Theorem 2.40. Consider the Hilbert space $\mathcal{H}_c := \oplus_{\phi\in\mathcal{S}(A)}\mathcal{H}_\phi$, where $\mathcal{P}(A)$ denotes the pure state space of A. The Hilbert space $\mathcal{H}_c$ naturally carries a representation $\pi = \oplus_{\phi\in\mathcal{P}(A)}\pi_\phi$. If $\pi(a) = 0$ then also $\pi(a^*a)\xi_\phi = 0$ for each $\phi \in \mathcal{P}(A)$. Since $\phi(a) = \langle \xi_\phi, \pi_\phi(a)\xi_\phi \rangle$ this implies $\phi(a^*a) = 0$. From this we can conclude [6] that $\sigma(a^*a) = \{0\}$, from what we can conclude $||a|| = 0$, thus $a = 0$. Consequently π is injective which proves Theorem 2.40. □

2.3 Von-Neumann algebras

In the following, we will investigate a special instance of a C^*-algebra, that frequently appears in mathematics as well as in physics, e.g., quantum field theory. The subject of operator algebras historically started with what is nowadays called von Neumann algebras, in honor of the founder of the subject.

2.3.1 The double commutant theorem

Definition 2.46. Let $\mathcal{H}$ be a Hilbert space. If A is a strongly closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, i.e., closed in the strong operator topology, we call A a von-Neumann algebra on $B(\mathcal{H})$.

Theorem 2.47 (Neumann). Let $A \subset B(H)$ be a unital, self-adjoint $*$-subalgebra of $B(\mathcal{H})$. Then the following conditions are equivalent

(1) A is a von Neumann algebra

(2) $A'' = A$

(3) A is closed in the weak operator topology

Proof. The implication (a) $\Rightarrow$ (b) follows directly and the equivalence (b) $\Leftrightarrow$ (c) follows from Corollary 2.36. It remains to prove (c) $\Rightarrow$ (a). For this consider for each $\xi \in \mathcal{H}$ the projection p of $\mathcal{H}$ onto $\overline{A\xi}$, the norm closure of the set $\{A\xi \,|\, \xi \in H\}$. For each $x \in A$, we then

[6] In order to make this conclusion, one needs a theorem of the following kind: Let A be a unital C^*-algebra. For any normal element $A \in A$ and $\lambda \in \sigma(a)$ there is a pure state $\phi \in \mathcal{P}(A)$ such that $\omega(a) = \lambda$. Since the proof typically relies on Gelfand-theory, i.e., the Gelfand isomorphism for commutative C^*-algebras what was not introduced so far, we omit the proof. We refer the reader to [34, 35, 36].

have $(xp)(\mathcal{H}) \subset p(\mathcal{H})$ whenever $(\mathbb{1} - p)xp = 0$. Choosing x to be self-adjoint, i.e., $x = x^*$, this implies that $xp = pxp$ is also self-adjoint whence $xp = px$. Since A is self-adjoint, it follows that $x \in A'$. Now let $y \in A''$. By definition, we have $yp = py$ and therefore $y\xi \in p(\mathcal{H})$, since $p\xi = \xi$, as A is assumed to be unital. For each $\epsilon > 0$ we can therefore find $x \in A$ such that $||y\xi - x\xi|| < \epsilon$. Hence, y can be approximated with elements from A on each single vector $\xi \in \mathcal{H}$. Now take $\xi_1, ..., \xi_n \in \mathcal{H}$ and define similar to the proof of Theorem 2.35 $\xi \in \mathcal{H}^n$ as the orthogonal sum of the ξ_k. By using $B(\mathcal{H}^n) \cong \mathrm{M}_n(B(\mathcal{H}))$ we see that

$$(r\tau(x) - \tau(x)r)_{kl} = r_{kl}x - xr_{kl} \tag{2.39}$$

for every $r = (r)_{kl} \in \mathrm{M}_n(B(\mathcal{H}))$ and $x \in B(\mathcal{H})$. From this it follows that the commutant of $\tau(A)$ in $\mathrm{M}_n(B(\mathcal{H}))$ consists of the matrices with entries in A', i.e., $\tau(A)' = M_n(A')$. If we now apply the first part of the proof with $\mathcal{H}^n, \tau(A)$ and $\tau(s)$ in place of $\mathcal{H}$, A and s, we obtain an element $x \in A$ such that

$$\sum_{k=1}^{n} ||(s - t)\xi_k||^2 = ||(\tau(s) - \tau(x))\xi||^2 < \epsilon^2 \tag{2.40}$$

Consequently, every $s \in A''$ can be approximated arbitrarily well in the strong topology with elements from A. Since A was assumed to be strongly closed, we have $s \in A$ and $A'' = A$. □

Example 2.48. Let Ω be a compact Hausdorff space and suppose that μ is a finite positive regular Borel measure on Ω, i.e., $\mu(A) = \sup\{\mu(K) \,|\, K \subset A\,,\, K \text{ compact}\}$ for all $A \in \mathfrak{B}(\Omega)$. We will show that

$$L^\infty(\Omega, \mu) \to B(L^2(\Omega, \mu)) \quad , \quad \varphi \mapsto M_\varphi \text{ where } M_\varphi(f) = \varphi f \tag{2.41}$$

for all $f \in L^2(\Omega, \mu)$ is an isometric $*$-homomorphism. That the map M_φ is bounded follows from

$$||M_\varphi(f)||_2^2 = \int_\Omega |\varphi(\omega)f(\omega)|^2 \,\mathrm{d}\mu(\omega) \leq ||\varphi||_\infty^2 \int_\Omega |f(\omega)| \,\mathrm{d}\mu(\omega) = ||\varphi||_\infty^2 ||f||_2^2 \tag{2.42}$$

By definition of the operator norm it follows from (2.42) that $||M_\varphi|| \leq ||\varphi||_\infty < \infty$. We will call M_φ the multiplication operator. Further the adjoint of M_φ is given by $M_{\overline{\varphi}}$ and thus the map $\varphi \mapsto M_\varphi$ is a $*$-homomorphism. We will now show that this map is also an isometry, i.e., we have $||\varphi||_\infty = ||M_\varphi||$. This can be proven by contradiction. So assume that this map is not an isometry and hence there exists $\epsilon > 0$ such that $||\varphi||_\infty - \epsilon > ||M_\varphi||$. This implies the existence of a Borel set $S \in \mathfrak{B}(\Omega)$ such that $\mu(S) > 0$ and $|\varphi(\omega)| > ||M_\varphi|| + \epsilon$ for all $\omega \in S$. Since μ is a regular measure, we can assume that S is compact. Using again the regularity of μ we also have $\mu(S) < \infty$. This implies

$$||M_\varphi^2||\, \mu(S) \geq ||M_\varphi \chi_S||_2^2 = \int_\Omega |\varphi(\omega)\chi_S(\omega)|^2 \,\mathrm{d}\mu(\omega) \geq \int_\Omega (||M_\varphi|| + \epsilon)^2 \chi_S(\omega) \,\mathrm{d}\mu(\omega) \tag{2.43}$$

Since $(||M_\varphi|| + \epsilon)^2$ is independent of ω and the integral of a simple function $\chi_S(\omega)$ is equal to the evaluation of the measure $\mu(S)$ we obtain after dividing by $\mu(S)$, that $||M_\varphi|| \geq ||M_\varphi|| + \epsilon$. But this is a contradiction to $\epsilon > 0$ and we have $||M_\varphi|| = ||\varphi||_\infty$. To sum up, we have shown that the map $\varphi \mapsto M_\varphi$ is an isometric $*$-isomorphism of $L^\infty(\Omega, \mu)$ onto a C^*-subalgebra of $B(L^2(\Omega, \mu))$. In particular, the space $\mathcal{C} = \{M_\varphi \,|\, \varphi \in L^\infty([0, 1], \lambda^1)\}$ is an abelian subalgebra of $B(L^2([0, 1], \lambda^1))$.

Theorem 2.49. Let $\mathcal{H}$ be a separable Hilbert space and let $A \subset B(\mathcal{H})$ be an abelian von Neumann algebra. Then $A = W^*(a)$ for some self-adjoint operator $a \in B(\mathcal{H})$, i.e., A is generated by a single element.

Theorem 2.50. A maximal self-adjoint operator $a \in B(\mathcal{H})$ is unitarily equivalent to the multiplication operator on $L^2(\sigma(a), \mu)$, where μ is a probability measure on the spectrum $\sigma(a) \subset \mathbb{R}$. In particular, the map $\mathfrak{B}(\sigma(a)) \to W^*(a)$ induces an isomorphism of von Neumann algebras

2.3.2 The classification of maximal abelian $*$-algebras in $B(\mathcal{H})$

In the following we are going to prove a theorem, which classifies all maximal abelian $*$-subalgebras in $B(\mathcal{H})$ up to unitary equivalence. As we will see, in contrast to the finite dimensional case where the maximal abelian subalgebra of $B(\mathcal{H})$ was simply the set of all diagonal matrices on $\mathcal{H}$ and therefore isomorphic to $\mathbb{C}^n$ with $n = \dim(\mathcal{H})$, the uniqueness is lost on the general case.

Definition 2.51. Let $\mathfrak{A}_1 = (X_1, \Sigma_1, \mu_1)$ and $\mathfrak{A}_2 = (X_2, \Sigma_2, \mu_2)$ be measure spaces.

(a) We call $\mathfrak{A}_1$ and $\mathfrak{A}_2$ equivalent if there exists a measurable bijection $\varphi : X_1 \to X_2$ with a measurable inverse and if the measures $\varphi_*\mu_1$ and μ_2 are equivalent in the sense that $\varphi_*\mu_1(A_2) = 0$ if and only if $\mu_2(A_2) = 0$ for each $A_2 \in \Sigma_2$. Here $\varphi_*\mu_1$ is a measure defined on (X_2, Σ_2) by virtue of

$$\varphi_*\mu_1(A_2) = \mu_1(\varphi^{-1}(A_2)) \quad \text{with} \quad A_2 \in \Sigma_2 \tag{2.44}$$

(b) We call $\mathfrak{A}_1$ and $\mathfrak{A}_2$ isomorphic, if there exists a measurable bijection $\varphi : X_1 \to X_2$ with measurable inverse and $\varphi_*\mu_1(A_2) = \mu_2(A_2)$ for all $A_2 \in \Sigma_2$.

Lemma 2.52. Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be measure spaces, where $K_1, K_2 \subset \mathbb{R}$ are compact subsets and Σ_i is a σ-algebras on K_i inherited from the Borel structure on $\mathbb{R}$ for $i = 1, 2$. Further μ_1, μ_2 are probability measures on (K_1, Σ_1) and (K_2, Σ_2) respectively and suppose that the measure spaces (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) are isomorphic. Then there exists a unitary operator such that

$$u : L^2(K_1, \mu_1) \to L^2(K_1, \mu_2) \quad \text{with} \quad uL^\infty(K_1, \mu_1)u^{-1} = L^\infty(K_2, \mu_2) \tag{2.45}$$

Proof. In the following we will assume that all appearing maps are measurable with respect to the Borel structure on $\mathbb{R}$. First recall that for an invertible map $\varphi : K_1 \to K_2$ the substitution formula is given by

$$\int_{K_2} f \circ \varphi^{-1} \,\mathrm{d}\,(\varphi_*\mu_1) = \int_{K_1} f \,\mathrm{d}\mu_1 \tag{2.46}$$

where $f : K_1 \to \mathbb{C}$. If φ is an isomorphism of measure spaces, (2.46) can we rewritten as

$$\int_{K_2} f \circ \varphi^{-1} \,\mathrm{d}\mu_2 = \int_{K_1} f \,\mathrm{d}\mu_1 \tag{2.47}$$

If $\varphi_*\mu_1$ and μ_2 are equivalent, they are also mutually absolutely continuous and thus the Radon-Nikodym derivatives $\mathrm{d}(\varphi_*\mu_1)/\mathrm{d}\,\mu_2$ and $\mathrm{d}(\varphi_*^{-1}\mu_2)/\mathrm{d}\,\mu_1$ exist. Using the Radon-Nikodym theorem and (2.47) one obtains that the operator

$$u : L^2(K_1, \mu_1) \to L^2(K_2, \mu_2) \quad \text{with} \quad L^2(K_1, \mu_1) \ni \xi \mapsto u\xi := \sqrt{\frac{\mathrm{d}(\varphi_*\mu_1)}{\mathrm{d}\mu_2}}\xi \circ \varphi^{-1} \tag{2.48}$$

is isometric. In addition, the map u also has an inverse given by

$$u^{-1}: L^2(K_2,\mu_2) \to L^2(K_1,\mu_1) \quad \text{with} \quad L^2(K_2,\mu_2) \ni \psi \mapsto \sqrt{\frac{\mathrm{d}(\varphi_*^{-1}\mu_2)}{\mathrm{d}\mu_1}}\psi\circ\varphi \tag{2.49}$$

what turns u into a unitary map. From Example 2.48 we already know that for $f \in L^\infty(K_1,\mu_1)$ we have $uM_f u^{-1} = M_{f\circ\varphi^{-1}}$. Further Example 2.48 tells us that the map $f \mapsto M_f$ injects $L^\infty(K_1,\mu_1)$ isometrically into $B(L^2(K_1,\mu_1)$ and the analog statement holds for $L^\infty(K_2,\mu_2)$. The map $f \mapsto f\circ\varphi^{-1}$ given then an isomorphism between $L^\infty(K_1,\mu_1)$ and $L^\infty(K_2,\mu_2)$. This follows from the fact that $||f\circ\varphi^{-1}||_\infty^{ess} = ||f||_\infty^{ess}$ what yields injectivity. The surjectivity follows then from the invertibility of φ, since $g \in L^\infty(K_2,\mu_2)$ is the image of $f = g\circ\varphi \in L^\infty(K_1,\mu_1)$. □

The next theorem is a deep and fundamental classification theorem in measure theory and goes back to the work of Kuratowski. In its general form it applies to polish spaces i.e., separable completely metrizable topological spaces. In the context of the classification of maximal abelian $*$-subalgebras if $B(\mathcal{H})$ it is sufficient to state it for probability measures on compact spaces $K \subset [0,1]$.

Theorem 2.53 (Kuratowski). Let (K,Σ,μ) be a probability space with $K \subset \mathbb{R}$ compact, Σ σ-algebra inherited from $\mathfrak{B}(\mathbb{R})$ such that $\mu(A) > 0$ for infinitely many $A \in \Sigma$. Then (K,Σ,μ) is isomorphic to one of the following possibilities:

(a) $([0,1],\mathfrak{B}([0,1]),\lambda^1)$

(b) $([0,1],\mathfrak{B}([0,1]),\mu)$ where $\mu = t\lambda^1 + (1-t)\nu$ and $t \in (0,1)$

(c) $([0,1],\mathfrak{B}([0,1]),\mu)$ where $\mu = t\lambda^1 + (1-t)\mu_n$, $t \in (0,1)$ and μ_n

(d) $K = \vec{N} = \{2^{-n} : n \in \mathbb{N}\} \cup \{1\}$ equipped with any probability measure ν for which $\nu(\{2^{-n}\}) > 0$ for each $n \in \mathbb{N}$ and $\nu(\{1\}) = 0$.

where μ_n is an arbitrary strictly nonzero probability measure on the n-point set $\vec{n} := \{\frac{1}{n}, ..., \frac{n-1}{n}, 1\}$.

Theorem 2.54. Let $\mathcal{H}$ be a separable Hilbert space and let $A \subset B(\mathcal{H})$ be a maximal abelian subalgebra. Then A is unitarily equivalent to one of the following objects:

(a) $L^\infty(0,1) \subset B(L^2(0,1))$

(b) $\ell^\infty(\mathbb{N}) \subset B(\ell^2(\mathbb{N}))$

(c) $L^\infty(0,1) \oplus \ell^\infty(\mathbb{N}) \subset B(L^2(0,1) \oplus \ell^2(\mathbb{N}))$

(d) $L^\infty(0,1) \oplus D_n(\mathbb{C}) \subset B(L^2(0,1)) \oplus \mathbb{C}^n$ for some $n \in \mathbb{N}$

Further all this possibilities are mutually unitarily inequivalent.

Proof. We first show that all this possibilities are mutually unitarily inequivalent. This can be proven by using the notion of atomic projections in a von Neumann algebra $M \subset B(\mathcal{H})$. For the situation here, we can easily classify the projections. The nonzero projections in $L^\infty([0,1])$ are the characteristic functions on measurable subsets of $[0,1]$ of positive Lebesgue measure. Since any such subset properly contains another such subset, there are no atomic projections in $L^\infty([0,1])$. The nonzero projections in $\ell^\infty(\mathbb{N})$ are given by the characteristic functions on $\mathbb{N}$, i.e., the one-dimensional projections δ_x for $x \in \mathbb{N}$. Hence $\ell^\infty(\mathbb{N})$ has countably many atomic projections. In particular, each projection majorizes an atomic one. In the same way, also $L^\infty(0,1) \oplus \ell^\infty(\mathbb{N})$ has countably many atomic projections, namely those which stem from $\ell^\infty(\mathbb{N})$ as well as uncountably many projections that do not majorize any atomic one. For the case of $L^\infty(0,1) \oplus D_n(\mathbb{C})$ first note that the atomic projections of $D_n(\mathbb{C})$ are the one-dimensional ones and thus $L^\infty(0,1) \oplus D_n(\mathbb{C})$ has exactly n atomic projections as well as uncountably many projections that do not majorize any atomic one, namely the ones in $L^\infty(0,1)$. Since unitary equivalence preserves the structure of atomic projections, each pair of elements from the list (a)-(d) cannot be unitarily equivalent. The remaining step is to prove that (a)-(d) already covers all possibilities. Due to Theorem 2.49 it is sufficient to consider abelian von Neumann algebras $A = W^*(a)$ for some a maximal. Further by virtue of Theorem 2.50 we can restrict to the case where a is the position operator on $L^2(K,\mu)$ where $K = \sigma(a) \subset \mathbb{R}$ is compact and μ is a regular probability measure with respect to the Borel structure inherited from $\mathbb{R}$ and $\mathrm{supp}(\mu) = K$ and therefore

$$W^*(a) = L^\infty(K,\mu) \subset B(L^2(K,\mu)) \tag{2.50}$$

We can now use Theorem 2.53 to obtain that one of the possibilities is given by $W^*(a) = ([0,1], \mathfrak{B}([0,1]), \lambda^1)$ what implies that $A \cong L^\infty(0,1) \subset B(L^2(0,1))$ and thus gives the first unitary equivalence (a). To obtain the second define $u : L^2(\vec{\mathbb{N}}, \nu) \to \ell^2(\mathbb{N})$ via $u\psi(n) = \sqrt{\nu(n)}\psi(2^{-n})$. This operator is unitary and it intertwines $UL^\infty(\vec{\mathbb{N}},\nu)u^{-1} = \ell^\infty(\mathbb{N})$. Therefore we obtain case (b). The cases (c), (d) can be obtained by the following construction. Denote by $\mathcal{A} \subset K$ the set of atoms in (K, Σ, μ). Clearly one can decompose K into disjoint subsets of the form

$$K = (K \setminus \mathcal{A} \bigsqcup \mathcal{A}) \tag{2.51}$$

For a given measure μ this gives rise to an orthogonal decomposition in the sense that $L^2(K,\mu) = L^2(K \setminus \mathcal{A}, \mu) \oplus L^2(\mathcal{A}, \mu)$. Further if we denote by $\pi = 1_{K\setminus\mathcal{A}}$ the projection onto the subset $K \setminus \mathcal{A}$ and $1_{L^2(K,\mu)} - \pi = 1_{\mathcal{A}}$ we can write $L^2(K \setminus \mathcal{A}, \mu) = \pi L^2(K,\mu)$ and $L^2(\mathcal{A},\mu) = 1_{\mathcal{A}} L^2(K,\mu)$. Since the measure μ can be decomposed into an atomic part μ_a and a continuous part μ_c this gives $L^2(K,\mu) = L^2(K,\mu_c) \oplus L^2(\mathcal{A},\mu_a)$ what induces a decomposition $L^\infty(K,\mu) = L^\infty(K,\mu_c) \oplus L^\infty(\mathcal{A},\mu_a)$, $L^\infty(K,\mu_c) = \pi L^\infty(K,\mu) = \pi L^\infty(K,\mu)\pi$ as well as $L^\infty(\mathcal{A},\mu_{\dashv}) = 1_{\mathcal{A}} L^\infty(K,\mu)$. This proves that $([0,1], \mathfrak{B}([0,1]), \mu)$ where $\mu = t\lambda^1 + (1-t)\nu$ and $t \in (0,1)$ from Theorem 2.53 yields (c). To obtain (d) consider the unitary map

$$u : L^2(\vec{n}, \mu_n) \to \mathbb{C}^n \quad \text{with} \quad u\psi_m = \sqrt{\mu_n(\frac{m}{n})}\psi(\frac{m}{n}) \text{ for } m = 1, ..., n \tag{2.52}$$

which yields the unitary equivalence $uL^\infty(\vec{n},\mu_n)u^{-1} = D_n(\mathbb{C})$. □

2.4 Frame-Theory

In the theory of vector spaces one of the most fundamental concepts is that of a basis. Given a basis for the vector space, one can write any element in the space in a unique way as a linear combination of elements in the basis. Thought the conditions to a basis are very restrictive, e.g., no linear dependence of the basis elements and in certain situations one also requires orthogonality. This makes it difficult or impossible to incorporate extra conditions to a given basis. The notion of a frame weakens these conditions and therefore provides a more flexible tool. In the notation and representation we mainly follow [41].

Definition 2.55. Let I be a at most countable index set. A family of elements $\{f_i\}_{i \in I} \subset \mathcal{H}$ is called a frame for $\mathcal{H}$ if there exists constants $A, B > 0$ such that

$$A||f||^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B||f||^2 \quad \forall f \in \mathcal{H} \tag{2.53}$$

The numbers A, B are called frame bounds. The optimal lower and upper frame bound is the supremum over all lower frame bounds and the optimal upper frame bound is the infimum over all upper frame bounds. The frame is called normalized, if $||f_i|| = 1$ for all $i \in I$.

Here it is important to note that the optimal frame bounds are actual frame bounds. Further, if $|I| = n < \infty$ the Cauchy-Schwarz inequality yields

$$\sum_{i=1}^{n} |\langle f, f_i \rangle|^2 \leq \left(\sum_{i=1}^{n} ||f_i||^2 \right) ||f||^2 =: B\,||f||^2 \tag{2.54}$$

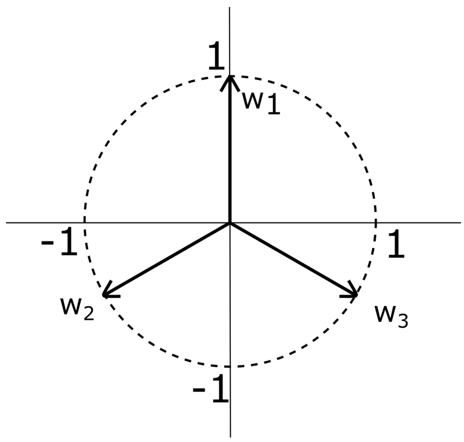


Figure 2.1: Configuration of vectors of the Mercedes frame. Each of the vectors is of length one, i.e., a member of the unit sphere S^1. Further they are located equiangular.

Definition 2.56. If only the right hand side of (2.53) is given, we call $(f_i)_{i \in I}$ a Bessel sequence with Bessel bound B. If $A = B$, we call it an A-tight frame and if $A = B = 1$ it is called a Parseval frame. If $||f_i|| = \alpha \in \mathbb{R}$ for all $i \in I$ it is an equal norm frame and if $||f_i|| = 1$ for all $i \in I$, it is a unit norm frame.

Obviously, every orthonormal basis for a Hilbert space $\mathcal{H}$ is a Parseval frame. If $(e_i)_{i \in I}$ denotes this basis and $f \in \mathcal{H}$ we have

$$\sum_{i \in I} |\langle f, e_i \rangle|^2 = \langle f, \sum_{i \in I} \langle f, e_i \rangle e_i \rangle = \langle f, f \rangle = ||f|| \tag{2.55}$$

Due to this equality we can choose $A = B = 1$ and obtain therefore a unit norm Parseval frame. However, not any Parseval frame is necessarily orthogonal nor a basis. For instance, consider the so called Mercedes frame, which is a frame for $\mathbb{R}^2$ consisting of three vectors w_1, w_2, w_3. The exact form of the vectors is given by $w_k = (\cos(\frac{2\pi}{3}(k-1)), \sin(\frac{2\pi}{3}(k-1)))$ for $k = 1, 2, 3$. Clearly, this is a unit norm frame which is also tight, as the upper and lower bound can be easily computed to $A = B = \frac{3}{2}$. A rescaling of the vectors w_k by $\alpha = \sqrt{(2/3)}$ yields a new frame (v_1, v_2, v_3) with $v_k = \alpha w_k$ which is a Parseval frame for $\mathbb{R}^2$, i.e., every vector $\xi \in \mathbb{R}^2$ satisfies

$$\xi = \langle \xi, u_1 \rangle u_1 + \langle \xi, u_2 \rangle u_2 + \langle \xi, u_3 \rangle u_3 \tag{2.56}$$

Note that the coefficients in the linear combination (??) are not unique, i.e., there may exist other coefficients κ_k different from $\langle \xi, u_k \rangle$, such that $\xi = \kappa_1 u_1 + \kappa_2 u_2 + \kappa_3 u_3$. However, it turns out that the frame representation $f = \sum_{i \in I} \langle f, f_i \rangle f_i$ shares useful stability properties. Apart from this easy example, one usually requires much larger sets of vectors that form a frame, often in very high dimensional spaces.

Theorem 2.57. Let $\{f_i\}_{i=1}^n \subset \mathcal{H}$ be a sequence. Then $\{f_i\}_{i=1}^n$ is a frame for $\text{span}\{f_i\}_{i=1}^n$.

Proof. Without loss of generality we can assume that not all f_i are zero. As we have already seen above, by choosing

$$B = \sum_{i=1}^n ||f_i|| \tag{2.57}$$

we can fulfill the upper frame condition. Denote by $W = \text{span}\{f_i\}_{i=1}^n$ and consider the mapping

$$\varphi : \mathcal{H} \to \mathbb{R} \quad , \quad f \mapsto \sum_{i=1}^n |\langle f, f_i \rangle|^2 \tag{2.58}$$

Recall that $\mathcal{H}$ is finite dimensional if and only if the unit ball is compact. Since $W \subset \mathcal{H}$, also the unit ball of W is compact and thus we can find a $g \in W$ with $||g||$ such that

$$A := \sum_{i=1}^n |\langle g, f_i \rangle|^2 = \inf \{ \sum_{i=1}^n |\langle f, f_i \rangle|^2 \ : \ f \in W \, , \, ||f|| = 1 \} \tag{2.59}$$

since φ reaches as a continuous function its maximum on a compact set. It is clear that for $f \in W = \text{span}\{f_i\}_{i=1}^n$ we have $A > 0$. As any $f \in W$ can be written as a rescaling of a unit vector we have for $0 \neq f \in W$

$$\sum_{i=1}^n |\langle f, f_i \rangle|^2 = \sum_{i=1}^n |\langle \frac{f}{||f||}, f_i \rangle|^2 ||f||^2 \geq A ||f||^2 \tag{2.60}$$

□

Corollary 2.58. Let $\{f_i\}_{i=1}^n \subset \mathcal{H}$ be a family of elements in $\mathcal{H}$. Then $\{f_i\}_{i=1}^n$ is a frame for $\mathcal{H}$ if and only if $\text{span}\{f_i\}_{i=1}^n = \mathcal{H}$.

By virtue of Corollary 2.58, a frame for a vector space $\mathcal{H}$ has to contain at least the same number of elements as a basis but also can contain more elements. To make this more precise, let $\{f_i\}_{i=1}^n$ be a frame for $\mathcal{H}$ and $\{g_j\}_{j=1}^m$ an arbitrary finite collection of elements of $\mathcal{H}$. Then also $\{f_i\}_{i=1}^n \cup \{g_j\}_{j=1}^m$ is a frame for $\mathcal{H}$. A frame which is not a basis is said to be overcomplete or redundant.

Definition 2.59. Let $\mathcal{H}$ be a Hilbert space and $(f)_{i\in I} \subset \mathcal{H}$ a Bessel sequence. The synthesis operator for $(f_i)_{i\in I}$ is the bounded linear operator $T : \ell_2(I) \to H$ given by $T(e_i) = f_i$ for all $i \in I$. The analysis operator for $(f_i)_{i\in I}$ is the adjoint T^* of the synthesis operator T. The frame operator of the frame $(f_i)_{i\in I}$ is the operator $S := TT^* : \mathcal{H} \to \mathcal{H}$.

Lemma 2.60. Let $(f_i)_{i\in I}$ be a frame for a Hilbert space $\mathcal{H}$ with frame bounds A, B.

(a) The frame operator S is bounded, invertible, self-adjoint and positive

(b) The family $(S^{-1}f_i)_{i\in I}$ is a frame with bounds B^{-1} and A^{-1}. If A, B are optimal bounds for $(f_i)_{i\in I}$, then the bounds A^{-1}, B^{-1} are optimal for $(S^{-1}f_i)_{i\in I}$. The frame operator for $(S^{-1}f_i)_{i\in I}$ is S^{-1}.

Proof. We first prove (a). As $(f_i)_{i\in I}$ is a Bessel sequence, the synthesis operator is bounded which turns the frame operator as the composition of bounded operators into an bounded operator, since $||S|| = ||TT^*|| = ||T||\,||T^*|| = ||T||^2 \leq B$. Further the self-adjointness follows from $S^* = (TT^*)^* = TT^* = S$. By assumption and definition of a frame, we have $A||f|| \leq \langle Sf, f\rangle \leq B||f||$ for all $f \in \mathcal{H}$ and thus $A\mathbb{1} \leq S \leq B\mathbb{1}$. Since $A > 0$, we have S positive. In addition it follows from $0 \leq \mathbb{1} - B^{-1}S \leq B^{-1}(B-A)\,\mathbb{1}$ that

$$||\mathbb{1} - B^{-1}S|| = \sup_{||f||=1} |\langle(\mathbb{1} - B^{-1}S)f, f\rangle| \leq \frac{B-A}{B} < 1 \tag{2.61}$$

which shows that S is invertible. It remains to show (b). If $f \in \mathcal{H}$ we have

$$\sum_{i\in I} |\langle f, S^{-1}f_i\rangle|^2 = \sum_{i\in I} |\langle S^{-1}f, f_i\rangle|^2 \leq B||S^{-1}f||^2 \leq B||S^{-1}||\,||f|| \tag{2.62}$$

Hence, also $(S^{-1}f_i)_{i\in I}$ is a Bessel sequence and the frame operator associated to this frame is well defined. This operator acts on $f \in \mathcal{H}$ as

$$\sum_{i\in I} \langle f, S^{-1}f_i\rangle S^{-1}f_i = S^{-1}\sum_{i\in I}\langle S^{-1}f, f_i\rangle f_i = S^{-1}SS^{-1}f = S^{-1}f \tag{2.63}$$

Consequently the frame operator of the frame $(S^{-1}f_i)_{i\in I}$ is equal to S^{-1}. Since the operator S^{-1} commutes with S and $\mathbb{1}$, a multiplication of $A\,\mathbb{1} \leq S \leq B\,\mathbb{1}$ with S^{-1} yields $B^{-1}\,\mathbb{1} \leq S^{-1} \leq A^{-1}\,\mathbb{1}$. □

Theorem 2.61. Let $(f_i)_{i\in I}$ be a frame for a Hilbert space $\mathcal{H}$ with frame operator S. Then we have

$$f = \sum_{i\in I}\langle f, S^{-1}f_i\rangle f_i \quad \forall\, f \in \mathcal{H} \tag{2.64}$$

and

$$f = \sum_{i\in I}\langle f, f_i\rangle S^{-1}f_i \quad \forall f \in \mathcal{H} \tag{2.65}$$

Both series (2.64) and (2.65) converge unconditionally for all $f \in \mathcal{H}$.

Proof. Let $f \in \mathcal{H}$ be arbitrary. By virtue of Lemma 2.60 we have

$$f = SS^{-1}f = \sum_{i\in I}\langle S^{-1}f, f_i\rangle f_i = \sum_{i\in I}\langle f, S^{-1}f_i\rangle f_i \tag{2.66}$$

Since $(f_i)_{i\in I}$ is a Bessel sequence and $(S^{-1}f_i)_{i\in I} \in \ell^2(\mathbb{N})$ we conclude that (2.64) converges unconditionally. The second claim (2.65) follows analog by replacing $f = SS^{-1}f$ by $f = S^{-1}Sf$. □

Chapter 3

The Formulation of the Problem

3.1 The Genesis

In 1959 Kadison and Singer posed the following fundamental question, that can be seen as a mathematical refinement of a claim which dates back to Dirac.

Question 3.1 ([27]). Does every pure state on the (abelian) von Neumann algebra $\ell^\infty(\mathbb{N})$ of bounded diagonal operators on $\ell_2(\mathbb{N})$ have a unique extension to a pure state on $B(\ell_2(\mathbb{N}))$, the von Neumann algebra of all bounded operators on $\ell_2(\mathbb{N})$?

The complete proof of the statement turns out to be difficult and relies on topology, operator theory and geometrical aspects of polynomials. In this section, we firstly want to understand why Question 3.1 is formulated in this particular way and why until 2014 it was the only remaining problem. As not only the solution of problem has a long history, but also the problem itself, any insight to the original problem leaded also to an adjustment of this question. However, the most natural as well as most naive way posing this question would be the following.

Question 3.2. Let $\mathcal{H}$ be a separable Hilbert space and let $A \subset B(\mathcal{H})$ be a commutative subalgebra. Has any pure state ω_A on A a unique extension ω to $B(\mathcal{H})$ such that ω can be chosen pure?

Theorem 3.3. If some abelian unital C^*-subalgebra $A \subset B(\mathcal{H})$ has the Kadison-Singer property, then A is necessarily maximal.

Proof. The idea behind the proof is to use the Gelfand isomorphism i.e., the relation $A \cong C(P(A))$ where $P(A) \subset S(A)$ is the pure state space of A. Suppose that A has the Kadison-Singer property and that $A \subset B \subset B(\mathcal{H})$ for some abelian unital C^*-algebra B. Consequently a pure state ω_A on A has then a unique pure extension ω on $B(\mathcal{H})$, which restricts to some state ω_B on B. We will first show that ω_B is also a pure state on B. Suppose the contrary, that is $\omega_B = t\omega_1 + (1-t)\omega_2$ and $\omega_1, \omega_2 \in S(B(\mathcal{H})))$. Since ω_A is pure by assumption we can conclude from $\omega_A = \omega_B|_A = t\omega_1|_A + (1-t)\omega_2|_A$ that $\omega_1|_A = \omega_2|_A = \omega_A$ and hence $\omega_1, \omega_2 \in S_A$. Since $\omega_A \in \partial_e S_A$, this implies $\omega_1 = \omega_2 = \omega_B$ and thus ω_B is pure. Abstractly this gives rise to a unique map $P(A) \to P(B)$ with $\omega_A \mapsto \omega_B$. The inverse of this map is given by the pullback of the inclusion $A \hookrightarrow B$, that is $\omega_B \mapsto \omega_B|_A$. All this together says that we have a bijection $P(A) \cong P(B)$. Since for any pair of C^*-algebras $A \subset B$ the pullback $S(B) \to S(A)$ is continuous in the w^*-topology, the map $\omega_B \mapsto \omega_A$ is continuous which implies that it is also a homeomorphism. Thus we have $A \cong B$ through the inclusion $A \hookrightarrow B$ what gives $A = B$ and A maximal. □

Therefore Theorem 3.3 forces one to readjust Question 3.2 in order to obtain a positive answer and therefore the refined version could be formulated as follows.

Question 3.4. Let $\mathcal{H}$ be a separable Hilbert space and let $A \subset B(\mathcal{H})$ be a maximal commutative subalgebra. Has any pure state ω_A on A a unique extension ω to $B(\mathcal{H})$ such that ω can be chosen pure?

Now we are in the comfortable situation that we have already classified all maximal commutative subalgebras in $B(\mathcal{H})$. In particular, Theorem 2.54 tells us that up to unitarily equivalence there only exist four cases, namely $L^\infty(0,1)$, $\ell^\infty(\mathbb{N})$, $D_n(\mathbb{C})$ for $n \in \mathbb{N}$ and direct sums thereof. As we will see in the following, the finite dimensional case $D_n(\mathbb{C})$ provides a positive answer, i.e., any pure state on $D_n(\mathbb{C})$ has a unique extension (c.f. Theorem 3.12) to $B(\mathbb{C}^n) \cong \mathrm{M}_n(\mathbb{C})$. However, the situation dramatically changes if we pass to $L^\infty(0,1)$. Here, already Kadison and Singer constructed a counterexample by using so called diagonal processes what can be seen as a generalization of taking the diagonal of an operator. We present the construction of this example in Theorem 3.19. Therefore we arrive at the following formulation of the problem.

Question 3.5 ([27]). Does every pure state on the (abelian) von Neumann algebra $\ell^\infty(\mathbb{N})$ of bounded diagonal operators on $\ell_2(\mathbb{N})$ have a unique extension to a pure state on $B(\ell_2(\mathbb{N}))$, the von Neumann algebra of all bounded operators on $\ell_2(\mathbb{N})$?

Thus we finally arrived at Question 3.1 which was posed by Kadison and Singer. As it turns out, we can formulate the problem in an even more precise way. To do so, we have to recall that the state space $S(B(\mathcal{H}))$ can be divided into two different classes of states, namely the normal and the singular states. That the Kadison Singer problem has a positive solution for normal states is the content of the following

Theorem 3.6. Let $\mathcal{H}$ be a separable Hilbert space and ω_A be a normal pure state on a maximal commutative unital C^*-algebra $A \subset B(\mathcal{H})$. Then ω_A has a unique extension to a state ω on $B(\mathcal{H})$, which is itself necessarily pure and normal.

Proof. First, we will use the fact (cf. II.5.5.17 in [39]) that any state ω admits a convex decomposition of the form $\omega = t\omega_n + (1-t)\omega_s$, where ω_n is a normal state, ω_s a singular state and $t \in [0,1]$. Hence if ω is pure, it must be either normal or singular. Since any normal state can be represented by virtue of a density operator, the possibility that ω_A is normal whereas ω is singular is excluded. Therefore ω must be normal and can be associated with a density operator ρ, i.e., $\omega(x) = \mathrm{tr}[\rho x]$ for $x \in B(\mathcal{H})$. The proof of the uniqueness is essentially the same as in Theorem 3.12. □

3.1.1 The finite dimensional case

To get some insight into the import structures and to see how the extension machinery works, we first consider the finite dimensional case, i.e., $\mathcal{H} \cong \mathbb{C}^n$. The first step is to understand the structure of commutative subalgebras of $B(\mathcal{H})$ and their ordering with respect to inclusion. For $a \in B(\mathcal{H})$ we write $C^*(a)$ for the C^*-algebra generated by a and $\mathbb{1}$, i.e., the algebra of all polynomials in a. We will mainly follow [48].

Theorem 3.7. If $a \in B(\mathcal{H})$ is self adjoint then $C^*(a)$ is commutative and the following properties hold.

(a) $C(\sigma(a)) \cong C^*(a)$ where the isomorphism $C(\sigma(a)) \ni f \mapsto f(a)$ is unique if it is subjected to the condition $(1_{\sigma(a)} : \lambda \mapsto 1) \mapsto \mathbb{1}$ and $(\mathrm{id}_{\sigma(a)} : \lambda \mapsto \lambda) \mapsto a$.

(b) If $\{e_\lambda\}$ is the set of spectral projections of the operator a we have

$$C^*(a) = C^*(e_\lambda \,|\, \lambda \in \sigma(a)) \tag{3.1}$$

(c) With respect to the isomorphism defined in (a) we have the identification $e_\lambda = \delta_\lambda(a)$ where δ_μ is a function on $\delta(a)$ defined by $\delta_\mu : \nu \mapsto \delta_{\mu\nu}$.

Proof. Let $a \in B(\mathcal{H})$ be self-adjoint. For each complex polynomial $p(x) : \mathbb{R} \to \mathbb{C}$ with $p(x) = \sum_k c_k x^k$ associate an operator via $p(a) := \sum_k c_k a^k$. Obviously, if p_1, p_2 are two polynomials and $\alpha \in \mathbb{C}$, we have $(\alpha p_1 + p_2)(a) = \alpha p_1(a) + p_2(a)$, $(p_1 p_2)(a) = p_1(a)p_2(a)$ and $p(a)^* = \overline{p}(a)$. Abstractly, we have defined an $*$-algebra homomorphism, or evaluation $\Phi_a : \mathbb{C}[x] \to C^*(a)$. Hence, the space $\Delta^*(a)$ of all such polynomials in a is a $*$-subalgebra of $B(\mathcal{H})$. In particular, as a linear subspace of $B(\mathcal{H})$, also $\Delta^*(a)$ is finite dimensional and consequently a C^*-algebra. For $p(x) = 1$, we obtain $\Phi_a(p) = a^0 = \mathbb{1}$, making $\Delta^*(a)$ unital. Choosing $p(x) = x$ implies $a \in \Delta^*(a)$ and therefore we have $\Delta^*(a) = C^*(a)$. To show the isomorphism property, define the map $C(\sigma(a)) \ni f \mapsto f(a)$. At first, suppose we restrict to polynomials p, such that $f(a) := p(a)$ as before. Since $C^*(a) = \Delta^*(a)$ consists of polynomials in a, the map $C(\sigma(a)) \to C^*(a)$ is surjective. To see that it is also injective, suppose that $p, q \in C(\sigma(a))$ such that $p(a) = q(a)$. If $\lambda \in \sigma(a)$ and $v_\lambda \in \mathcal{H}_\lambda$, where $\mathcal{H}_\lambda$ is the eigenspace with respect to λ, we obtain

$$q(a) = p(a) \Leftrightarrow q(a)v_\lambda = p(a)v_\lambda \, \forall \lambda \in \sigma(a) \Leftrightarrow q(\lambda)v_\lambda = p(\lambda)v_\lambda \, \forall \lambda \in \sigma(a) \tag{3.2}$$

and we can conclude that $p = q$ as functions on $\sigma(a)$. If now $f \in C(\sigma(a))$ is an arbitrary function, there exist some polynomial p that coincides with f on $\sigma(a) \subset \mathbb{R}$, such that $f(a)$ can be defined via $p(a)$. The remaining two claims can be proven by using the orthogonality relation $e_\mu e_\lambda = \delta_{\mu\nu} e_\lambda$ for the spectral projections. Since any polynomial in a is also self-adjoint, we obtain by the spectral theorem for self-adjoint endomorphisms

$$f(a) = \sum_{\lambda \in \sigma(a)} f(\lambda) e_\lambda \tag{3.3}$$

Now define $G(a) := \text{span}(\{e_\lambda \,|\, \lambda \in \sigma(a)\} \cup \{\mathbb{1}\})$. Clearly by the properties of the spectral projections, $G(a)$ itself is a commutative C^*-algebra. In particular, by (3.3) we have $C^*(a) \subset G(a)$ as well as $G(a) \subset C^*(a)$ and thus $C^*(a) = G(a)$ what proves the claim. □

Definition 3.8. Let $\vec{a} = (a_1, ..., a_n)$ be commuting self-adjoint operators.

(a) A vector $v \in \mathcal{H} \backslash \{0\}$ is called a joint eigenvector of $\vec{a} = (a_1, ..., a_n)$ such that $a_i v = \lambda_i v$ with $\lambda_i \in \mathbb{C}$ for each $i \in \{1, ..., n\}$. We also write $\vec{a}v = \vec{\lambda}v$ and call $\vec{\lambda}$ a joint eigenvalue of $\vec{a}$.

(b) The joint spectrum $\sigma(a_1, ..., a_n) =: \sigma(\vec{a})$ consists of all joint eigenvalues of $\vec{a}$

(c) $C^*(\vec{a})$ is the smallest unital C^*-subalgebra of $B(\mathcal{H})$ that contains each a_i.

Clearly we have $\sigma(\vec{a}) \subset \sigma(a_1) \times ... \times \sigma(a_n) \subset \mathbb{R}^n$. In general, this inclusion will be strict. For example consider the case of $a_1 = \sigma_1 \otimes \sigma_1, a_2 = \sigma_2 \otimes \sigma_2$ and $a_3 = \sigma_3 \otimes \sigma_3$, where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices, i.e., the canonical generators of $SU(2)$ [1]. Since $\sigma(\sigma_i) = \{-1, 1\}$, by the tensor product structure we have $\sigma(a_i) = \{-1, 1\}$. However, there is only one joint eigenvector.

[1]Note that the set of commuting operators is part of the so called Peres-Mermin square, a construction to demonstrate quantum contextuality, i.e., the property, that the outcome of a measurement does not only depend on the state of the system, but also on compatible measurements that are simultaneously performed.

Lemma 3.9. Let $\vec{a} = (a_1, ..., a_n)$ be a family of commuting self-adjoint operators on $\mathcal{H}$. Then there is a self-adjoint operator $a \in B(\mathcal{H})$ such that $C^*(\vec{a}) = C^*(a)$.

Proof. Define the operator

$$a := \sum_{\vec{\lambda} \in \sigma(\vec{a})} c_{\vec{\lambda}} e_{\vec{\lambda}} \tag{3.4}$$

such that $c_{\vec{\lambda}}$ are different from each other. Therefore we can conclude by using Theorem 3.7, that $C^*(a) = C^*(e_{\vec{\lambda}} \,|\, \vec{\lambda} \in \sigma(\vec{a}))$ and the claim follows. □

Lemma 3.10. Every unital commutative C^*-algebra $A \subset B(\mathcal{H})$ is generated by a single self-adjoint operator a and the unit $\mathbb{1}$. In particular, any unital commutative C^*-algebra $A \subset B(\mathcal{H})$ is of the form $A = C^*(a)$.

Proof. Since A is a C^* algebra it is in particular also a $\mathbb{C}$-vector space. Therefore we can find a basis $(\xi_k) \subset A$ of A and we can decompose $\xi_k = a_k + ib_k$ with a_k, b_k self-adjoint. Note that the operators can be constructed easily via $a_k = \frac{1}{2}(\xi_k + \xi_k^*)$ and $b_k = \frac{1}{2}(\xi_k - \xi_k^*)$. Since we assume that A is commutative, we must have $\xi_k \xi_k^* = \xi_k^* \xi_k$ and hence ξ_k is a normal operator. This is equivalent to the commutativity of a_k and b_k □

As we have already seen for the case of arbitrary separable Hilbert spaces, a unital commutative C^*-algebra $C \subset B(\mathcal{H})$ is called maximal, if it is not contained in some bigger unital commutative C^*-algebra contained in $B(\mathcal{H})$. In the context of finite dimensional Hilbert spaces we will call an operator $a \in B(\mathcal{H})$ maximal, if $|\sigma(a)| = n = \dim(\mathcal{H})$. In other words, $a \in B(\mathcal{H})$ is maximal if each eigenvalue is nondegenerate. Note that $C^*(a)$ is maximal if and only if a is maximal. To see this, assume the converse, i.e., that there exists $\kappa \in \sigma(a)$ with multiplicity $m_\kappa > 1$. Thus the corresponding spectral decomposition e_κ can be decomposed into $e_\kappa = e_\kappa^1 + e_\kappa^2$ where e_κ^i are orthogonal projections and therefore commute. We could then extend $C^*(a)$ to $C^*(e_\lambda, e_{\kappa_1}, e_{\kappa_2})$ and we obtain a contradiction to the maximality of $C^*(a)$.

Theorem 3.11. A unital commutative C^*-algebra $A \subset B(\mathcal{H})$ is maximal if and only if it is unitarily equivalent to the algebra $D_n(\mathbb{C})$ of all diagonal matrices.

Proof. Assume that $\dim(\mathcal{H}) = n$. Then $D_n(\mathbb{C})$ is indeed a maximal abelian subalgebra in $\mathrm{M}_n(\mathbb{C}) \cong B(\mathcal{H})$. If there would exist a proper extension of $D_n(\mathbb{C})$ then it has to contain some additional matrix $b \in \mathrm{M}_n(\mathbb{C})$ which commutes with all $a \in D_n(\mathbb{C})$. This operator b must then fulfill $(db)_{ij} = d_{ii} b_{ij} = b_{ij} d_{jj} = (bd)_{ij}$ for all $d \in D_n(\mathbb{C})$ what forces $b \in D_n(\mathbb{C})$. By virtue of Lemma 3.10 any unital commutative C^* algebra A can be written as $A = C^*(a)$ for an appropriate self-adjoint operator $a \in B(\mathcal{H})$ also the maximal one is of this form, i.e., $C = C^*(a)$. In order to make $C^*(a)$ maximal we have to choose a maximal in which case we can describe a uniquely by the spectrum $\sigma(a) = \{\lambda_1, ..., \lambda_n\}$ and the corresponding eigenvectors $\{v_{\lambda_1}, ..., v_{\lambda_n}\}$. This naturally induces a unitary map $u : \mathcal{H} \to \mathbb{C}^n$ with $uv_{\lambda_j} = u_j$ and therefore we have $uau^{-1} = \mathrm{diag}(\lambda_1, ..., \lambda_n)$. Since a is self-adjoint and maximal, all λ_j are real and different any vector $z = (z_1, ..., z_n) \in \mathbb{C}^n$ can be written as $z_i = p(\lambda_i)$ for some suitable polynomial $p \in \mathbb{C}[x]$. Hence $uC^*(a)u^{-1} = D_n(\mathbb{C})$. □

Theorem 3.12. Let $\mathcal{H}$ be a finite dimensional Hilbert space and let ω_A be a pure state on a maximal commutative unital C^*-algebra $A \subset B(\mathcal{H})$. Then ω_A has a unique extension to a state ω on $B(\mathcal{H})$ which is necessary pure.

Proof. By virtue of Theorem 3.11 we can assume that $\mathcal{H} = \mathbb{C}^n$ and that A consists of all diagonal matrices. Since we can regard any vector in $\mathbb{C}^n$ as the diagonal of a matrix in $\mathrm{M}_n(\mathbb{C})$ we have $D_n(\mathbb{C}) \cong \mathbb{C}^n$. From this we can conclude that the action of ω_A on some $b \in D_n(\mathbb{C})$ is given by $\omega_A(b) = b_j$ for some $j = 1, ..., n$. Since any state on a finite dimensional space is normal, the extension must, in the case of existence, be given by $\omega(a) = \mathrm{tr}[\rho a]$ with a suitable unique operator $\rho \in D_n(\mathbb{C})$. If we denote the eigenvectors of ρ by v_i and if u_i denotes the standard basis, we have

$$\sum_i p_i |\langle u_j, v_i \rangle|^2 = 1 \tag{3.5}$$

Since $\sum_{\lambda \in \sigma(\rho)} \lambda = 1$ and $|\langle u_j, v_i \rangle| \leq 1$, we conclude that (3.5) can only hold for a given j if $|\langle u_j, v_i \rangle| = 1$ for all i with $p_i > 0$. Therefore we obtain that $\rho = v_i v_i^*$ which proves that ρ exists, is unique and pure. □

3.1.2 The case $L^\infty(0, 1)$

The aim of this section is to show, that states on $L^\infty(0, 1)$ do not have the Kadison Singer property [18]. The proof relies on the concept of diagonal processes, introduced by von Neumann [6], which can be seen as on operation to take the diagonal part of an operator relative to a maximal abelian self-adjoint subalgebra. Further, with the help of this abstract machinery, also the result for the finite dimensional case can be reproduced.

Definition 3.13 ([6]). For any bounded operator A and any projection operator E we define

$$A^{|E} := EAE + (1 - E)A(1 - E) \tag{3.6}$$

Further, if $E_1, ..., E_n$ are projections, we write $A^{|E_1|\cdots|E_n}$ for the iterated version of (3.6).

As already mentioned, the process $A^{|E_1|E_2|\cdots}$ should be seen as an analogue of the process of taking the diagonal of a finite matrix. To see that Definition 3.13 reproduces the diagonal part for the case $B(\mathcal{H}) = \mathrm{M}_2(\mathbb{C})$. If we set $E_1 := |0\rangle\langle 0|$ and $E_2 := |1\rangle\langle 1|$ we obtain with $\mathbb{1} - |1\rangle\langle 1| = |0\rangle\langle 0|$

$$A^{|E_1|E_2} = |1\rangle\langle 1| A^{|E_1} |1\rangle\langle 1| + |0\rangle\langle 0| A^{|E_1} |0\rangle\langle 0| = \langle 0|A|0\rangle |0\rangle\langle 0| + \langle 1|A|1\rangle |1\rangle\langle 1| \tag{3.7}$$

what exactly gives the diagonal of the matrix A. By induction, it is easy to show that if $B(\mathcal{H}) = \mathrm{M}_n(\mathbb{C})$ and $E_i = |i\rangle\langle i|$ is an orthonormal basis, then

$$A^{|E_1|E_2\cdots|E_n} = \sum_{i=1}^{n} \langle i|A|i\rangle |i\rangle\langle i| \tag{3.8}$$

However, if the range of projection operators is not one-dimension but still $E_i E_j = 0$ for $i \neq j$ and the family is a resolution of the identity, one obtain a matrix in block form.

Lemma 3.14. Let A be a bounded operator and $E, E_1, ..., E_n$ a family of projections. Then the followings statements hold.

(a) $A^{|E}$ commutes with E

(b) If $E_1, ..., E_n$ pairwise commute and $\pi \in \mathfrak{S}_n$ then

$$A^{|E_1|E_2|\cdots|E_n} = A^{|E_{\pi(1)}|E_{\pi(2)}|\cdots|E_{\pi(n)}} \tag{3.9}$$

(c) If $E_1, ..., E_n$ pairwise commute, then $A^{|E_1|\cdots|E_n}$ commutes with all of them

(d) $||A^{|E_1|\cdots|E_n}|| \leq ||A||$

Proof. To (a): We have $A^{|E}E = EAE + (1-E)A(1-E)E = EAE$, since $(1-E)E = 0$. Since the same is true for $EA^{|E}$, both expressions are equal to EAE and thus they commute. To (b): Since any permutation can be written as a concatenation of transpositions, it is sufficient to consider the case where π is the transposition of two neighbours, say m and $m+1$. Define $B := A^{|E_1|\cdots|E_{m-1}}$. If we can prove $B^{|E_m|E_{m+1}} = B^{|E_{m+1}|E_m}$ and then apply the process $|E_{m+2}|\cdots|E_n$ the claim follows. Hence it is sufficient to show $A^{|E|F} = A^{|F|E}$ for two commuting projections E, F. Since $[E, F] = 0$ implies $[(1-E), (1-F)] = 0$ a direct calculation yields

$$\begin{aligned} A^{|E|F} &= FA^{|E}F + (1-F)A^{|E}(1-F) = FEAEF + F(1-E)A(1-E)F \\ &+ (1-F)EAE(1-F) + (1-F)(1-E)A(1-E)(1-F) = A^{|F|E} \end{aligned} \tag{3.10}$$

To (c): Let $i \in \{1, ..., n\}$ and let $\pi \in \mathfrak{S}_n$ such that $\pi(n) = i$. By (b) we have $A^{|E_1|\cdots|E_n} = A^{|E_{\pi(1)}|\cdots|E_{\pi(n)}}$ and this commutes by (a) with $E_{\pi(n)} = E_i$. To (d): We will prove the more general result that $||A^{|E_1|\cdots|E_n}|| \leq ||A^{|E_1|\cdots|E_{n-1}}||$. If we define $B := A^{|E_1|\cdots|E_{n-1}}$ and $E := E_n$, we obtain $||B^{|E}|| \leq ||B||$. Therefore we have to prove that for any $f, g \in \mathcal{H}$ we have

$$|\langle B^{|E}f, g\rangle| \leq ||B|| \, ||f|| \, ||g|| \tag{3.11}$$

By definition we have

$$\langle B^{|E}f, g\rangle = \langle BEf, Eg\rangle + \langle B(1-E)f, (1-E)g\rangle \tag{3.12}$$

By using Cauchy-Schwarz inequality we obtain

$$|\langle Bf, g\rangle| \leq ||A|| \left(||Ef|| \, ||Eg|| + ||(1-E)f|| \, ||(1-E)g||\right) \tag{3.13}$$

Applying Cauchy-Schwarz again at the second factor we can further estimate

$$\begin{aligned} &||Ef|| \, ||Eg|| + ||(1-E)f|| \, ||(1-E)g|| \\ &\leq \left[\left(||Ef||^2 + ||(1-E)f||^2\right)\left(||Eg||^2 + ||(1-E)g||^2\right)\right]^{\frac{1}{2}} \end{aligned} \tag{3.14}$$

Now it is easy to see that $||Ef||^2 + ||(1-E)f||^2 = \langle Ef, Ef\rangle + \langle(1-E)f, (1-E)f\rangle = \langle f, f\rangle = ||f||^2$. Similar we obtain $||Eg||^2 + ||(1-E)g||^2 = ||g||^2$. Together this implies

$$||Ef|| \, ||Eg|| + ||(1-E)f|| \, ||(1-E)g|| \leq ||f|| \, ||g|| \tag{3.15}$$

Inserting (3.15) into (3.12) yields (3.11). □

Lemma 3.15 ([18]). Let $M \subset B(\mathcal{H})$ be an abelian von Neumann algebra generated by projections $\{E_i\}_{i \in \mathbb{N}}$. If $p \in \beta(\mathbb{N}) \setminus \mathbb{N}$, then there exists a linear operator $\phi_p : B(\mathcal{H}) \to B(\mathcal{H})$, such that

(a) for all $A \in B(\mathcal{H})$ and all $n \in \mathbb{N}$ we have $[\phi_p(A), E_n] = 0$, i.e., $\phi_p(B(\mathcal{H})) \subset M'$. Further, $\phi_p(A)$ is weak closure point of the set of operators $\{A^{|E_1|\cdots|E_n}\}$ for each $A \in B(\mathcal{H})$.

(b) $\phi_p(AB) = A\phi_p(B)$ for each $A \in M$ and $B \in B(\mathcal{H})$.

(c) $\phi_p(\mathbb{1}) = \mathbb{1}$ and $\phi_p(A) \geq 0$ if $A \geq 0$.

Proof. With respect to the set of projection operators $\{E_i\}_{i\in\mathbb{N}}$ define the function

$$f : \mathbb{N} \to B(\mathcal{H}) \quad , \quad n \mapsto f(n) := A^{|E_1|\cdots|E_n} \tag{3.16}$$

By Lemma 3.14 (d) we have $||A^{|E}|| \leq ||A||$ and hence f maps $\mathbb{N}$ into the weakly compact ball with radius $||B||$ around $0 \in B(\mathcal{H})$. By Theorem 2.8, we know that f has a unique extension f_1 from $\mathbb{N}$ to $\beta(\mathbb{N})$, i.e., $f_1 : \beta(\mathbb{N}) \to B(\mathcal{H})$ such that $f_1(n) = f(n)$ for all $n \in \mathbb{N}$. In particular, f_1 is continuous and its range is also contained in ball of radius $||A||$ around the origin. Now define $\phi_p(A)$ to be $f_1(p)$. By definition, we have $(\alpha + B)^{|E} = \alpha A^{|E} + B^{|E}$ and since $\mathbb{N}$ is a dense subset of the T_2-space $\beta(\mathbb{N})$, we obtain the linearity of the map ϕ_p. Moreover, this implies that $\phi_p(A)$ is a weak closure point of $\{A^{|E_1|\cdots|E_n}\}$. For the case $A = \mathbb{1}$, we clearly have $\mathbb{1}^{|E_1|\cdots|E_n} = \mathbb{1}$ for all $n \in \mathbb{N}$ and thus $\phi_p(1) = 1$. If $A \geq 0$, we know that we can find $X \in B(\mathcal{H})$ such that $A = XX^*$. Therefore we have

$$A^{|E} = EXX^*E + (1-E)XX^*(1-E) = YY^* + ZZ^* \tag{3.17}$$

with $Y := EX$ and $Z := (1-E)X$. Since the sum of positive operators is positive, we can conclude that $A^{|E}$ is positive and hence also $A^{|E_1|\cdots|E_n}$ for all $n \in \mathbb{N}$. Consequently each weak closure point of $\{A^{|E_1|\cdots|E_n}\}$ is positive and thus also $\phi_p(A)$. It remains to show that $\phi_p(A) \in M'$ for all $A \in B(\mathcal{H})$. For a given n, we know that $\phi_p(A)$ is a weak closure point of $\{A^{|E_1|\cdots|E_m}\}$ for $m \geq n$ and each of which commutes with E_n. So, $\phi_p(A)$ commutes with E_n for each n and consequently $\phi_p(A) \in M'$. □

Definition 3.16. Let $M \subset B(\mathcal{H})$ be an abelian von Neumann algebra. A linear order preserving mapping $\phi : B(\mathcal{H}) \to M'$ is called a diagonal process relative to M if $\phi|_M = \mathrm{id}_M$. The diagonal process is called proper, if for all $A \in B(\mathcal{H})$ the image $\phi(A)$ is a weak closure point of the operators $A^{|E_1|\cdots|E_n}$ with $E_1, ..., E_n \in M$. Otherwise, it is called improper.

Although a diagonal process is defined via a weak closure of sequence elements, for certain special cases we can directly calculate its image. Namely, if ϕ is a proper diagonal process relative to M and $A \in M'$, then we know by Lemma 3.15 that $\phi(A)$ is a weak closure point of the operators $\{A^{|E_1|\cdots|E_n}\}$. But we have $A^{|E_1|\cdots|E_n} = A$ for all $n \in \mathbb{N}$ and thus $\phi(A) = A$. However, if the diagonal process fulfills certain continuity conditions, it turns out to be unique.

Lemma 3.17. Let $M \subset B(\mathcal{H})$ be an abelian von Neumann algebra, $\{E_n\}$ a generating family of projections for M and ϕ a diagonal process relative to M. If ϕ is weakly continuous on the unit ball, then it is the unique proper diagonal process relative to M and $\phi(A)$ is the weak limit of $\{A_n\}$ where $A_n := A^{|E_1|\cdots|E_n}$.

Proof. Suppose that ϕ is weakly continuous on the unit ball and thus on each bounded ball. Further assume that $\tilde{\phi}$ is a proper diagonal process relative to M. By definition, $\tilde{\phi}$ must be a weak closure point of $\{A^{|E_1|\cdots|E_1}\}$. Is now T such a weak closure point, we have that $\tilde{\phi}(T)$ is a weak closure point of $\{\phi(A^{|E_1|\cdots|E_n})\} = \{\phi(A)\}$. So we have $\phi(T) = \phi(A)$. Further, if $T \in M'$, we have already argued that $\phi(T) = T$ in the case of ϕ proper. Thus we have $\tilde{\phi}(A) = \phi(A)$ and consequently $\tilde{\phi} = \phi$. Since ϕ coincides with any proper diagonal process $\tilde{\phi}$ relative to M, it must be unique. Further, if T is a weak limit point of the sequence $(A^{|E_1|\cdots|E_n})$ we have $T \in M'$ since it commutes with each E_k. In addition it is a weak closure point of $\{A^{|E_1|\cdots|E_n}\}$, so that $T = \phi(A)$. By Lemma 3.15, we know that

$\{A^{|E_1|\cdots|E_n}\}$ is contained in the weakly compact ball of radius $||A||$ around $0 \in B(\mathcal{H})$ and hence we can conclude that the sequence $(A^{|E_1|\cdots|E_n})$ has a limit point which must be $\phi(A)$. Therefore $\phi(A)$ is the weak limit of $(A^{|E_1|\cdots|E_n})$ what completes the proof. □

We have already seen that $\ell^\infty(\mathbb{N})$ can be identified with the set of diagonal operators acting on a Hilbert space $\mathcal{H}$ that admits a countable basis. Let $\{x_k\}$ be an orthonormal basis for $\mathcal{H}$. We define the diagonal process ϕ for a bounded operator A as the operator whose matrix representation relative to the basis $\{x_k\}$ is the diagonal matrix with the diagonal of the matrix representation for A relative to $\{x_k\}$. Then ϕ is indeed a diagonal process relative to $\ell^\infty(\mathbb{N})$. If $x = \sum_k \alpha_k x_k$ and $||A|| \leq 1$ we obtain

$$|\langle \phi(A)x, x\rangle| \leq \sum_k |\alpha_k|^2 |\langle D(A)x_k, x_k\rangle| = \sum_k |\alpha_k|^2 |\langle Ax_k, x_k\rangle| \tag{3.18}$$

Since we assumed the operator A to be bounded, we know that for $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$ we have $\sum_k |\alpha_k|^2 |\langle Bx_k, x_k\rangle| < \frac{\epsilon}{2}||x||$. Together this yields

$$|\langle \phi(A)x, x\rangle| \leq \sum_k |\alpha_k|^2 \, |\langle Ax_k, x_k\rangle| < \epsilon \tag{3.19}$$

Consequently ϕ is a continuous map at $0 \in B(\mathcal{H})$ and by linearity it follows, that ϕ is continuous on the unit ball of $B(\mathcal{H})$ with respect to the weak operator topology. Therefore, we can conclude by Lemma 3.17, that ϕ is the unique proper diagonal process relative to $\ell^\infty(\mathbb{N})$. That it is not only the unique proper diagonal process relative to $\ell^\infty(\mathbb{N})$ but also the unique diagonal process relative to $\ell^\infty(\mathbb{N})$ is the content of the following

Corollary 3.18 ([18]). The unique diagonal process relative to $\ell^\infty(\mathbb{N})$ is ϕ.

Proof. Suppose that there are two distinct diagonal processes ϕ and $\tilde{\phi}$. Then there must exist $A \in B(\mathcal{H})$ such that $\phi(A) \neq \tilde{\phi}(A)$. In particular, we have $\langle \tilde{\phi}(A)x_k, x_k\rangle \neq \langle \phi(A)x_k, x_k\rangle$ for some $k \in \mathbb{N}$, what implies $\omega_{x_k} \circ \phi \neq \omega_{x_k} \circ \tilde{\phi}$. Note that both are valid state extensions of ω_{x_k} from $\ell^\infty(\mathbb{N})$ to $B(\mathcal{H})$. Since ω_{x_k} is a vector pure state of $\ell^\infty(\mathbb{N})$, it has a unique extension to $B(\mathcal{H})$. Therefore we must have $\omega_{x_k} \circ \phi = \omega_{x_k} \circ \tilde{\phi}$, what implies $\phi = \tilde{\phi}$. □

Theorem 3.19 ([18]). There is more than one proper diagonal process relative to $L^\infty(0,1) \subset B(L^2(0,1))$. In particular, pure state extension is not unique with respect to $L^\infty(0,1)$.

Proof. As we have already pointed out in Theorem 2.54, we can identify the continuous maximal abelian subalgebra of $B(\mathcal{H})$ with the algebra of multiplication operators $L^\infty(0,1)$ and therefore the set of projections $\{P_{km} \,|\, m \in \mathbb{N}, k = 1, ..., m\}$ corresponding to multiplication with the characteristic function on the closed intervals $[(k-1)/m, k/m]$ generate the continuous maximal abelian subalgebra. Clearly, we have for any $m \in \mathbb{N}$, that $\mathbb{1}_{[0,1]} = \sum_{k=1}^m E_{km}$. Further, for fixed m the intervals are disjoint what implies that they are pairwise orthogonal. In fact, we have

$$A^{|E_{1m}|E_{2m}|\cdots|E_{mm}} = \sum_{k=1} E_{km} A E_{km} \tag{3.20}$$

as well as for $n \in \mathbb{N}$ arbitrary

$$A^{|E_{1m}|\cdots|E_{mm}|E_{1(mn)}|\cdots|E_{(mn)(mn)}} = \sum_{k=1}^{mn} E_{k(mn)} A E_{k(mn)} \tag{3.21}$$

If there exists a unique diagonal process ϕ of the form ϕ_p with $p \in \beta(\mathbb{N})\setminus\mathbb{N}$, by Lemma 3.15 we know that $\phi(A)$ is the weak limit with respect to j of $\sum_{k=1}^{m} E_{km}AE_{km}$ with $m = 2^j$. If this is not the case for a certain bounded operator A, then $\phi_p(A) \neq \phi_{\tilde{p}}(A)$ for some points $p, \tilde{p} \in \beta(\mathbb{N}) \setminus \mathbb{N}$ with $p \neq \tilde{p}$. The remaining proof consists of the construction of such an operator A. First we introduce an orthonormal basis for the Hilbert space $\mathcal{H}$. The set of functions $\{f_k\}_{k\in\mathbb{Z}}$ with $f_k(x) := e^{2\pi ikx}$ turns out to be such a basis. The operator A is taken to be the projection Π onto the subspace $\Delta = \text{span}(\{f_{n_j} | j \in \mathbb{N}\})$ where the concrete form of the elements f_{n_j} will become evident during the proof. Consider

$$\begin{aligned} \langle E_{km}\Pi E_{km}(1), 1\rangle &= \sum_{j=1}^{\infty} |\langle f_{n_j}, E_{km}(1)\rangle|^2 = \sum_{j=1}^{\infty} \Big| \int_{\frac{k-1}{m}}^{\frac{k}{m}} e^{2\pi i n_j x}\, \mathrm{d}x \Big|^2 \\ &= \sum_{j=1}^{\infty} \Big|\frac{1}{2\pi i n_j}\left[e^{2\pi i n_j k/m} - e^{2\pi i n_j (k-1)/m}\right]\Big|^2 = \sum_{j=1}^{\infty} \frac{1}{4\pi^2 n_j^2}|e^{2\pi i n_j/m} - 1|^2 \end{aligned} \tag{3.22}$$

Since the expression (3.22) is independent of the particular $k \in \{0, ..., m\}$ we obtain

$$\langle \left(\sum_{k=1}^{m} E_{km}\pi E_{km}\right)(1), 1\rangle = m\sum_{j=1} \frac{m}{4\pi^2 n_j^2}|e^{2\pi i n_j/m} - 1|^2 = \sum_{j=1}^{\infty} \frac{m}{\pi^2 n_j^2} \sin\left(\frac{\pi n_j}{m}\right) \tag{3.23}$$

where we have used $|e^{2\pi ix} - 1|^2 = 2 - e^{2\pi ix} - e^{-2\pi ix} = 4\sin(\pi x)$. We are now going to show that for a suitable choice of $(n_j)_{j\in\mathbb{N}}$ the sequence

$$\frac{m}{\pi^2}\sum_{j=1}^{\infty} \frac{\sin^2(\frac{\pi n_j}{m})}{n_j^2} = \frac{1}{\pi}\sum_{j=1}^{\infty} \left[\frac{\pi n_j}{m}\right]^{-2} \sin^2\left(\frac{\pi n_j}{m}\right)\frac{\pi}{m} =: a_m \tag{3.24}$$

does not tend to a limit as $2^r =: n \to \infty$. The set $\{n_j\}$ is chosen[2] as

$$\{n_j\} = \bigcup_{k=1}^{\infty} [2^{2^k-2}, 2^{2^k-1}] \cap \mathbb{N} \tag{3.25}$$

The benefit of introducing the special sequence of (a_m) in (3.24) is that it can be rewritten as an integral over $\mathbb{R}_+$, where the integrand is given by a step function s_m defined as

$$s_m(x) := \begin{cases} a_m & \text{if } x \in [\frac{\pi(n_j-1)}{m}, \frac{\pi n_j}{m}] \\ 0 & \text{else} \end{cases} \tag{3.26}$$

for $j \in \mathbb{N}$. Is $m = 4\eta$ with $\eta \in \mathbb{N}$, we obtain

$$\frac{1}{\pi}\sum_{\frac{m}{4}<n_j\leq\frac{m}{2}} \left(\frac{m}{\pi n_j}\right)^2 \frac{\pi}{m}\sin^2(\frac{\pi n_j}{m}) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} s_m(x)\,\mathrm{d}x =: b_m \tag{3.27}$$

If we now define $m_k := 2^{2^k}$, then $(s_{m_k})_k$ becomes a Riemann approximating step function to $f(x) := \pi^{-1}x^{-2}\sin^2(x)$ on the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$.Therefore, if a_{2^m} tends to a limit as $m \to \infty$ so does the subsequence a_{m_k} for $k \to \infty$ and thus

$$\lim_{m\to\infty} a_{2^m} = \lim_{k\to\infty} a_{m_k} \geq \lim_{k\to\infty} b_{m_k} = \pi^{-1}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x^{-2}\sin^{-2}(x)\,\mathrm{d}x > \pi^{-2} \tag{3.28}$$

[2] As an illustrative example how the set looks like, we will calculate the first eight elements. For $k = 1, 2, 3$ we have $[1, 2]$, $[4, 8]$, $[64, 128]$. Consequently, the first elements of the sequence are $n_1 = 1, n_2 = 2, n_3 = 4, n_4 = 5, n_5 = 6, n_6 = 7, n_7 = 8, n_8 = 64$.

The last inequality in (3.28) can be seen as follows. The derivative of $2x^{-2}\sin(x)$ is given by $2x^{-3}\sin(x)\,[x\cos(x)-\sin(x)]$ which turns out to be negative on $[\frac{\pi}{4},\frac{\pi}{2}]$ (cf. Figure 3.1). From $2x^{-2}\sin(x)|_{x=(1/2)\pi}=\frac{4}{\pi^2}$ we can conclude that the values of the function are lower bounded on $[\frac{\pi}{4},\frac{\pi}{2}]$ by $\frac{4}{\pi^2}$, since the function is decreasing. All this together yields $4\pi^{-1}(\frac{\pi}{2}-\frac{\pi}{4})\pi^{-2}=\pi^{-2}$. On the other hand, there are no n_k in $(2^{2^k-1},2^{2^{k+1}-2})$. Consequently, if we

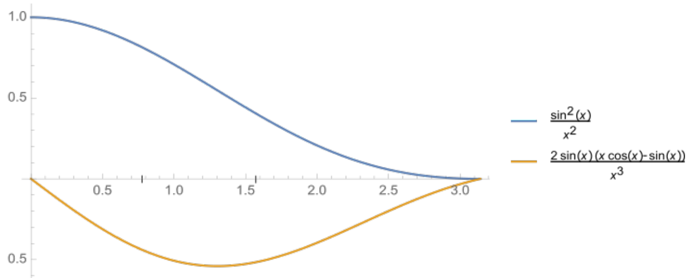


Figure 3.1: The relation between the function $f(x)=x^{-2}\sin^2(x)$ and its derivative. The interval $[\frac{\pi}{4},\frac{\pi}{2}]$ is indicated by two black lines. The derivative is clearly negative.

set $r_k:=2^{2^{k+1}-2}$, the step function s_{r_k} is zero on the interval $[\pi 2^{2^k}(2^{2^{k+1}-2})^{-1},\pi k(k+1)^{-1}]\xrightarrow{k\to\infty}[0,\pi]$. Hence, if we suppose that $\lim_{m\to\infty}a_{2^m}$ exists, we obtain

$$\pi^{-2}<\lim_{k\to\infty}a_{m_k}=\lim_{k\to\infty}a_{r_k}=\lim_{k\to\infty}\int_0^\infty s_{r_k}(x)\,\mathrm{d}x\le\frac{1}{\pi}\lim_{k\to\infty}\int_\pi^\infty x^{-2}\,\mathrm{d}x=\pi^{-2}\qquad(3.29)$$

what yields a contradiction. Note that we have used, that $s_m(x)\le\pi^{-1}x^{-2}$ on the interval $[0,\infty)$. Therefore $\lim_k a_{2^k}$ do not exist, and Π does not have a unique diagonal part relative to $L^\infty(0,1)$, if the basis elements are chosen accordingly to the constructed set $\{n_j\}$. Accordingly to the discussion at the beginning of Section 3.1.2, there are pure states of $L^\infty(0,1)$ which do not have a unique extension to all bounded operators. □

As already discussed, non-uniqueness of diagonal processes implies non-uniqueness of pure state extension. However, the uniqueness of the diagonal process does not lead to the uniqueness of pure state extension, i.e., the extension of a state must not be necessarily the concatenation of the state with a diagonal process. More precise, the uniqueness of the process is only necessary, but not sufficient. Consequently, the obtained results leave the question of uniqueness of state extension open for $\ell^\infty(\mathbb{N})$, or by using Lemma 3.12, the uniqueness of state extension of the singular pure states of $\ell^\infty(\mathbb{N})$. Kadison and Singer inclined to the view that also $\ell^\infty(\mathbb{N})$ does not have the unique extension property. Although not settling the problem completely, Reid [19] showed in a paper about representation of the Calkin algebra (cf. the originally paper of Calkin [49]), that certain singular pure states, i.e., states which correspond to points in $\beta(\mathbb{N})\setminus\mathbb{N}$, have the unique extension property.

Theorem 3.20 ([19]). If the ultrafilter $\mathcal{U}$ is rare, then the corresponding pure state of $\ell^\infty(\mathbb{N})$ has a unique state extension to $B(\mathcal{H})$.

A different and more general approach was taken by Anderson [22], who gave a characterization of those pure states on $\mathcal{B}\subset\mathcal{A}$, with $\mathcal{A}$ and $\mathcal{B}$ C^*-algebras, that admit a unique extension to $\mathcal{A}$ in terms of so called compressions.

Definition 3.21. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}\subset\mathcal{A}$ a C^*-subalgebra. If ω is a state on $\mathcal{A}$ define the set

$$\mathfrak{G}_\omega:=\{A\in\mathcal{A}\,|\,|\omega(A)|=||A||=1\}\qquad(3.30)$$

Is further φ a state on $\mathcal{B}$, we say that $\mathcal{A}$ is $\mathcal{B}$-compressible modulo φ if for each $A \in \mathcal{A}$ and each $\epsilon > 0$ there exists a $B \in \mathfrak{G}_\varphi$ and $Y \in \mathcal{B}$ such that $||BAB - Y|| < \epsilon$.

Theorem 3.22 ([22]). Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B} \subset \mathcal{A}$ a C^*-subalgebra. If ω is a pure state on $\mathcal{B}$, then ω has a unique pure state extension to $\mathcal{A}$ if and only if $\mathcal{A}$ is $\mathcal{B}$-compressible modulo ω. More precise, if $\tilde{\omega}$ is the unique pure state extension of ω to $\mathcal{A}$, then for each $\epsilon > 0$ there exists $B \in \mathfrak{G}_\omega$, such that $||B(X - \tilde{\omega}(X)\mathbb{1})B|| < \epsilon$. If $\mathcal{B}$ is a von Neumann algebra, the operator B can be taken to be a projection.

Theorem 3.23 ([22]). Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B} \subset \mathcal{A}$ a maximal abelian C^*-subalgebra. Then $\mathcal{B}$ has the extension property relative to $\mathcal{A}$ if and only if $\mathcal{A}$ is $\mathcal{B}$-compressible.

3.2 The Metamorphosis

3.2.1 Anderson's Infinite Paving Conjecture

Definition 3.24. Let $T \in B(\ell_2(I))$ with I at most countable. We say that T has (r, ϵ)-paving if there exists a partition $\{A_1, ..., A_r\}$ of I such that

$$||\Pi_{A_j} T \, \Pi_{A_j}|| \leq \epsilon \, ||T|| \quad \text{for } j = 1, ..., r \tag{3.31}$$

with Π_A the orthogonal projection of $\ell_2(I)$ onto the closure of span$\{e_i | i \in A\}$, where $\{e_i\}_{i \in I}$ standard basis of $\ell_2(I)$,

Conjecture 3.25 (Paving). For $\epsilon > 0$, there exists a natural number r such that for every $n \in \mathbb{N}$ and any $T \in B\left(\ell_2(I)\right)$ whose matrix has zero diagonal, we can find a (r, ϵ)-paving of T, i.e., we can find a partition $\{A_i\}_{i=1}^r$ of $\{1, ..., |I|\}$, such that

$$||\Pi_{A_j} T \Pi_{A_j}|| \leq \epsilon ||T|| \tag{3.32}$$

Lemma 3.26. Let $P \in \ell^\infty(I)$ be a diagonal projection and $\rho \in S(\ell^\infty(I))$ pure. Then we have $\rho\left(P\right) = 0$ or $\rho\left(P\right) = 1$.

Proof. To procure a contradiction, suppose that $\rho\left(P\right) = \lambda \in (0, 1)$. Since ρ is a state, we have $\rho\left(\mathbb{1} - P\right) = \rho\left(\mathbb{1}\right) - \rho\left(P\right) = 1 - \lambda$. Define the linear functionals $\rho_1, \rho_2 : \ell^\infty(I) \to \mathbb{C}$ by virtue of

$$A \mapsto \rho_1\left(A\right) := \frac{1}{\lambda}\rho\left(PA\right) \quad , \quad A \mapsto \rho_2\left(A\right) := \frac{1}{1-\lambda}\rho\left(\left(\mathbb{1} - P\right)A\right) \tag{3.33}$$

As one can easily check, ρ_1 as well as ρ_2 are states. Indeed

$$\rho\left(A\right) = \rho\left(\left(\frac{1-\lambda}{1-\lambda}(\mathbb{1} - P) + \frac{\lambda}{\lambda}P\right)A\right) = \lambda\rho_1 + \left(1 - \lambda\right)\rho_2 \tag{3.34}$$

Thus ρ cannot be a pure state. Contradiction. □

Theorem 3.27. The Infinite Paving Conjecture implies a positive solution to the Kadison-Singer Problem.

Proof. We have to show that by assuming the correctness of the Infinite Paving conjecture one can conclude that any pure state on $\ell^\infty(I)$ has a unique extension to $B(\ell_2(I))$. Clearly the trivial option is $\tilde{\omega}(T) := \omega(\text{diag}(T))$. In order to prove that this is the unique choice any extension $\tilde{\omega}$ must satisfy $\tilde{\omega}(T) = \omega(\text{diag}(T)) = \tilde{\omega}(\text{diag}(T))$ or equivalently $\tilde{\omega}(T - \text{diag}(T)) = 0$ for every $T \in B(\ell_2(I))$. Suppose now that ω is a pure state on $\ell^\infty(I)$ and let $\tilde{\omega}$ be an extension of it to $B(\ell_2(I))$. For $T \in B(\ell_2(I))$ and $\tilde{T} := T - \text{diag}(T)$ and $\epsilon > 0$ fixed let $\Pi_{A_1}, ..., \Pi_{A_k}$ be an ϵ-paving of $\tilde{T}$. It follows that

$$\tilde{\omega}(\tilde{T}) = \tilde{\omega}(\mathbb{1}\tilde{T}\mathbb{1}) = \sum_{i,j}^{k} \tilde{\omega}(\Pi_{A_i}\tilde{T}\,\Pi_{A_j}) \tag{3.35}$$

Since $A_i \cap A_j = \emptyset$, it follows that $\Pi_{A_i} + \Pi_{A_j}$ is also a diagonal projection and by the linearity of $\tilde{\omega}$ we have $\tilde{\omega}(\Pi_{A_i} + \Pi_{A_j}) = \tilde{\omega}(\Pi_{A_i}) + \tilde{\omega}(\Pi_{A_j})$. By virtue of Lemma 3.26 we know that exactly one of the projection, that we will then label Π_{A_1} satisfies $\tilde{\omega}(\Pi_{A_1}) = 1$ and $\tilde{\omega}(\Pi_{A_j}) = 0$ for all $2 \le j \le r$. Using Cauchy-Schwarz inequality, i.e., $|\omega(g^* f)|^2 \le \omega(f^* f)\omega(g^* g)$ we obtain that each term in the sum of (3.35) satisfies

$$|\tilde{\omega}(\Pi_{A_i}\tilde{T}\Pi_{A_j})| \le \sup\{\tilde{\omega}(\Pi^*_{A_i}\Pi_{A_i})\tilde{\omega}(\Pi^*_{A_j}\tilde{T}^*\tilde{T}\Pi_{A_j}), \tilde{\omega}(\Pi^*_{A_j}\tilde{T}^*\tilde{T}\Pi_{A_j})\tilde{\omega}(\Pi^*_{A_j}\Pi_{A_j})\} \tag{3.36}$$

Since all Π_{A_i} are projectors we can conclude, that only the first term in (3.35) is nonzero and thus we have $\tilde{\omega}(\tilde{T}) = \tilde{\omega}(\Pi_{A_1}\tilde{T}\,\Pi_{A_1}) \le ||\Pi_{A_1}\tilde{T}\,\Pi_{A_1}|| \le \epsilon\,||\tilde{T}||$ by assuming the paving property. Since $\epsilon > 0$ was chosen arbitrary, we can conclude that $\tilde{\omega}(\tilde{T}) = 0$. □

3.2.2 Anderson's Finite Paving Conjecture

Lemma 3.28. [50] Let $r \in \mathbb{N}$ be fixed and assume that for any $n \in \mathbb{N}$ there is a partition $\{A^n_j\}^r_{j=1}$ of the set $\{1, ..., n\}$. Then there exist natural numbers $\{k_1 < k_2 < ...\}$ such that if $m \in A^{k_m}_j$ for some $1 \le j \le r$ we have $m \in A^{k_l}_j$ for all $l \ge m$. Thus, if $A_j = \{m \,|\, m \in A^{k_m}_j\}$ then

(a) $\{A_j\}^n_{j=1}$ is a partition of $\mathbb{N}$

(b) If $A_j = \{m_1 < m_2 < ...\}$, then for all $l \in \mathbb{N}$ we have $\{m_1, m_2, ..., m_l\} \subset A^{k_{m_l}}_j$

Proof. We will prove the claim by induction. For each $n \in \mathbb{N}$, obviously $1 \in A^n_j$ for some $j = 1, ..., r$. Therefore, for every $n \in \mathbb{N}$ we can define j_n as the index such that $1 \in A^n_{j_n}$. Since this yields an infinite sequence $(j_i)_{i \ge 0}$ with $1 \le j_i \le r$ for each i, there exists at least one $1 \le j \le r$ that occurs infinitely often. Taking the smallest such index, we produce an increasing sequence of natural numbers $n^1_1 < n^1_2 < ...$ such that $1 \in A^{n^1_i}_j$. More precise, we have generated a sequence of partitions $(\{A^{n_i}_j\})_{i \ge 1}$ with $n_i \in \mathbb{N}$ and $n_i < n_{i+1}$. Similar, for every n^1_i, we have that $2 \in A^{n^1_i}_j$ for some $1 \le j \le r$. With the same argument as above we obtain that there exists a subsequence $(n^2_i)_{i \ge 1}$ of $(n^1_i)_{i \ge 1}$ and a $1 \le j \le r$ such that $2 \in A^{n^2_i}_j$. By induction, we obtain for all $l \in \mathbb{N}$ a subsequence $(n^{l+1}_i)_{i \ge 1}$ of the already constructed sequence $(n^l_i)_{i \ge 1}$ and a $1 \le j \le n$ such that $l + 1 \in A^{n^{l+1}_i}_j$ for all $i \in \mathbb{N}$. Define now $k_i = n^i_i$ for all i what proves the claim. □

Theorem 3.29. [50] The Infinite Pacing Conjecture is equivalent to the Finite Paving Conjecture.

Proof. We first show that the finite version implies the infinite one. Let $(a_{ij})_{i,j=1}^{\infty}$ be a bounded linear operator on $\ell_2(\mathbb{N})$ and fix $\epsilon > 0$. By assumption we can find for any given $n \in \mathbb{N}$ a partition $\{A_j^n\}_{j=1}^r$ of $\{1, ..., n\}$ such that if we define $T_n = (a_{ij})_{i,j=1}^n$ we have

$$||\Pi_{A_j^n} T_n \Pi_{A_j^n}|| \leq \frac{\epsilon}{2} ||T_n|| \leq \frac{\epsilon}{2} ||T|| \quad \text{for all } j = 1, ..., r \tag{3.37}$$

Let now $\{A_j\}_{j=1}^r$ be the partition of $\mathbb{N}$ constructed in Lemma 3.28. For fixed $1 \leq j \leq 1$ let $A_j = \{m_1 < m_2 < ...\}$ and for all $l \in \mathbb{N}$ let $\Pi_l = \Pi_{I_l}$ where $I_l = \{m_1, ..., m_l\}$. For $f \in \ell_2(\mathbb{N})$ and $l \in \mathbb{N}$ sufficient large we have

$$||\Pi_{A_j} T \Pi_{A_j}(f)|| \leq 2||\Pi_l \Pi_{A_j} T \Pi_{A_j} \Pi_l(f)|| = 2||\Pi_l \Pi_{A_j}^{k_{m_l}} T_{k_{m_l}} \Pi_{A_j}^{k_{m_l}} \Pi_l(f)|| \tag{3.38}$$

$$\leq 2||\Pi_{A_j}^{k_{m_l}} T_{k_{m_l}} \Pi_{A_j}^{k_{m_l}}|| \, ||\Pi_l(f)|| \leq 2\frac{\epsilon}{2} ||T|| \, ||f|| = \epsilon ||T|| \, ||f|| \tag{3.39}$$

Since (3.38) holds for arbitrary $f \in \ell_2(\mathbb{N})$, we can conclude that $||\Pi_{A_l} T \Pi_{A_l}|| \leq \epsilon ||T||$. Conversely assume that the infinite paving conjecture is true. We will prove the statement by a contradiction. Assume that (3.32) fails. This implies the existence of $\epsilon > 0$, a partition $\{I_n\}_{n=1}^{\infty}$ of $\mathbb{N}$ into finite subsets and a family of operators $T_n : \ell_2(I_n) \to \ell_2(I_n)$ such that $||T_n|| = 1$ and for every partition $\{A_k^n\}_{k=1}^n$ of I_n there exists a $k \in \{1, ..., n\}$ for which

$$||Q_{A_k^n} T_n Q_{A_k^n}|| \geq \epsilon \tag{3.40}$$

holds. Now define a new operator by

$$T := \bigoplus_{n=1}^{\infty} T_n : \bigoplus_{n=1}^{\infty} \ell_2(I_n) \to \bigoplus_{n=1}^{\infty} \ell_2(I_n) \tag{3.41}$$

By construction we have $||T|| = \sup_n ||T_n|| = 1$. We can then find a partition $\{A_k\}_{k=1}^r$ of $\mathbb{N}$ such that for all $k \in \{1, ..., r\}$ we have $||Q_{A_k} T Q_{A_k}|| \leq \epsilon$. For $n \in \mathbb{N}$ and $k = 1, ..., r$ define $A_k^n := A_k \cap I_n$. Since $\{A_k\}_{k=1}^r$ is a partition of $\mathbb{N}$, we have that $\{A_k^n\}_{k=1}^r$ is a partition of I_n. Thus we have

$$||Q_{A_k^n} T_n Q_{A_k^n}|| = ||Q_{A_k^n} T Q_{A_k^n}|| \leq ||Q_{A_k} T Q_{A_k}|| \leq \epsilon \tag{3.42}$$

yielding a contradiction to the assumption (3.40) if $n \geq r$. □

Corollary 3.30. The Finite Paving Conjecture implies a positive solution to the Kadison-Singer Problem.

3.2.3 Weaver's conjecture

The Weaver conjecture is an equivalent formulation of the Kadison-Singer problem, but settled in linear algebra. In particular, it is a reformulation in terms of a combinatorial problem about balanced sets of vectors in $\mathbb{C}^d$.

Definition 3.31. Let $v_1, ..., v_n \in \mathbb{C}^d$ for $d \in \mathbb{N}$ and $u \in \mathbb{C}^d$ a unit vector. The moment of the collection of vectors $v_1, ..., v_n$ in the direction of u is given by

$$M(u; v_1, ..., v_n) := \sum_{k=1}^{n} |\langle v_k, u \rangle|^2 \tag{3.43}$$

A collection of vectors $v_1, ..., v_n$ is called spherical, if the moment in (3.43) is independent of the particular direction $u \in \mathbb{C}^d$.

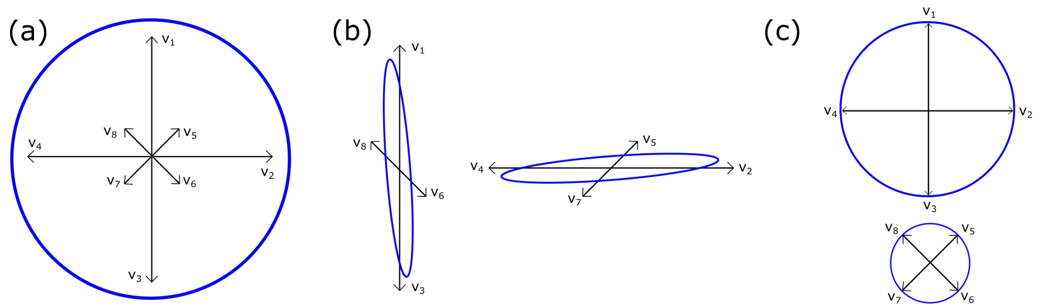


Figure 3.2: Illustration of the moment of vectors and Weaver's conjecture. The blue line is image of the map $S^1 \ni u \mapsto M(u; v_1, ..., v_8)$. (a) Collection of 8 vectors in a plane, where four among of them have an equal large norm, while the remaining have an equal small norm. The whole collection of vectors is spherical. (b) Partition of the vectors into two subsets ($r = 2$), whereat each of the sets contains two vectors with large norm and two vectors with small norm. Clearly, the moment of the partitions is not spherical. (c) Partition of the vectors into sets, where each set contains only those vectors that share the same norm. The resulting collections of vectors have a spherical moment.

Conjecture 3.32 ([51] , KS$_r$). There exist universal constants $N \geq 2$ and $\epsilon > 0$ such that the following holds. Let $v_1, ..., v_n \in \mathbb{C}^k$ satisfy $||v||_2 \leq 1$ for all i and suppose

$$\sum_i |\langle u, v_i \rangle|^2 \leq N \tag{3.44}$$

for every unit vector $u \in \mathbb{C}^k$. Then there exists a partition $X_1, ..., X_r$ of $\{1, ..., n\}$ such that

$$\sum_{i \in X_j} |\langle u, v_i \rangle|^2 \leq N - \epsilon \tag{3.45}$$

for every unit vector $u \in \mathbb{C}^k$ and all j.

Theorem 3.33 ([51]). The Kadison-Singer problem has a positive solution if and only if Conjecture KS$_r$ is true for some $r \geq 2$.

Proof. We prove that KS$_r$ is equivalent to the paving conjecture. Assume that there exist r, N and ϵ such that KS$_r$ holds. We have to show that this implies for every orthogonal projection $Q \in \mathrm{M}_n(\mathbb{C})$ with $\delta(Q) \leq 1/N$ the existence of diagonal projections $\Pi_1, ..., \Pi_r \in \mathrm{M}_n(\mathbb{C})$ which such up to the identity $\mathbb{1}_n$ and satisfy $||\Pi_j Q \Pi_j|| \leq 1 - \epsilon/N$ for all j. To do so, let $Q \in \mathrm{M}_n(\mathbb{C})$ be an orthogonal projection with $\delta(Q) \leq 1/N$. If rk$(Q) = k$ then its range is a k-dimensional subspace $V \subset \mathbb{C}^n$. Define $v_i = \sqrt{N} Q e_i \in V$ for $i = 1, ..., n$. Consequently one has for all i

$$||v||_2^2 = N||Pe_i||_2^2 = N\langle Pe_i, e_i \rangle = NP_{ii} \leq N\delta(P) \leq 1 \tag{3.46}$$

where we used that $\langle Pe_i, Pe_i \rangle = e_i^* P^* P e_i = e_i^* P^2 e_i = e_i^* P e_i = \langle Pe_i, e_i \rangle$. Furthermore, if $u \in V$ is a unit vector we have that $Pu = u$ and since P is self-adjoint we have

$$\sum_i |\langle u, v_i \rangle|^2 = \sum_i |\langle u, \sqrt{N} P e_i \rangle|^2 = N \sum_i |\langle u, e_i \rangle|^2 = N \tag{3.47}$$

Hence we fulfill the requirements of conjecture KS_r and KS_r asserts the existence of a partition $X_1, ..., X_n$ of $\{1, ..., n\}$ such that

$$\sum_{i\in X_j} |\langle u, v_i\rangle|^2 \leq N - \epsilon \tag{3.48}$$

for every unit vector $u \in V$ and all $j = 1, ..., r$. We now have to construct r diagonal projections $Q_1, ..., Q_r$ which sum up to the identity and satisfy $||Q_j P Q_j|| \leq 1 - \epsilon/N$ for all j. For this purpose let $Q_j \in \mathrm{M}_n(\mathbb{C})$ be defined by

$$Q_j : \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C}) \quad Q_j e_i = \begin{cases} e_i & \text{if } i \in X_j \\ 0 & \text{if } i \notin X_j \end{cases} \tag{3.49}$$

Since $\{X_i\}_{i=1}^r$ is a partition of $\{1, ..., n\}$ we have that $Q_1 + ... + Q_r = \mathbb{1}_n$ and for any unit vector $u \in V$ it follows

$$||Q_j P u||_2^2 = \sum_{i=1}^n |\left(Q_j P u\right)_i|^2 = \sum_{i=1}^n |\langle Q_j P u, e_i\rangle|^2 = \sum_{i=1}^n |\langle u, P Q_j e_i\rangle|^2 \tag{3.50}$$

where the third equality follows from $P^* = P$ and $Q_j^* = Q_j$. Further (3.46) and (3.49) yield

$$\sum_{i=1}^n |\langle u, P Q_j e_i\rangle|^2 = \sum_{i\in X_j} |\langle u, P e_i\rangle|^2 = \frac{1}{N}\sum_{i\in X_j} |\langle u, v_i\rangle|^2 \leq 1 - \epsilon/N \tag{3.51}$$

Since $u \in V$ was chosen arbitrarily, this shows that $||Q_j P Q_j|| = ||Q_j P \left(Q_j P\right)^*|| = ||Q_j P||^2 \leq 1 - \epsilon/N$.
For the other implication we show the contraposition. For this, suppose that the conjecture KS_r fails for all r. Fix $N = r \geq 2$ and let $v_1, ..., v_n \in \mathbb{C}^k$ be a counterexample with $\epsilon = 1$. For $w_i = v_i/\sqrt{N}$ one has

$$||A_{w_i}|| = \max_{\xi\in\mathbb{C}\setminus\{0\}} \frac{||A_{w_i}\xi||}{||\xi||} = \max_{\xi\in\mathbb{C}\setminus\{0\}} |\langle \xi, w_i\rangle| \frac{||w_i||}{||\xi||} = ||w_i||_2^2 = \frac{1}{N}||v_i||_2^2 \leq 1/N \tag{3.52}$$

Since $\sum_i A_{w_i} \leq \mathbb{1}_k$, we have that $\mathbb{1}_k - \sum_{i=1}^n A_{w_i} \geq 0$ with finite rank. In particular, we can find further positive rank one operators A_{w_i} with $i = n+1, ..., m$ such that $||A_{w_i}|| \leq 1/N$ and that $\{A_{w_j}\}_{j=1}^m$ is a partition of unity. Now define

$$\Phi : \mathbb{C}^k \hookrightarrow \mathbb{C}^m \quad , \quad u \mapsto \Phi(u) \;\text{ with }\; \left(\Phi(u)\right)_j = \langle u, w_j\rangle \tag{3.53}$$

for $j = 1, ..., m$. For $u \in \mathbb{C}^k$ we have

$$||\Phi u||_2^2 = \sum_{i=1}^m |\langle \Phi u, e_i\rangle|^2 = \sum_{i=1}^m |\langle u, w_i\rangle|^2 = \sum_{i=1}^m \langle A_{w_i} u, u\rangle = \langle \sum_{i=1}^m A_{w_i} u, u\rangle = ||u||_2^2 \tag{3.54}$$

Since Φ is an isometry and $\mathrm{im}(\Phi) \subset \mathbb{C}^m$ is a linear subspace, we can construct a unique orthogonal projection in $\mathbb{C}^{n+m}$ with range $\Phi\left(\mathbb{C}^k\right)$, i.e.,

$$\langle P e_i, \Phi w_j\rangle = \langle e_i, \Phi w_j\rangle = \langle w_i, w_j\rangle = \Phi w_i \tag{3.55}$$

for all i and all j. Consequently we have $Pe_i = \Phi w_i$ since $\{w_i\}$ span $\mathbb{C}^k$. Let $D := \text{diag}(P)$ i.e., D is a diagonal matrix with $d_{ii} = p_{ii}$ and thus $||D|| = \max_i ||w_i||_2^2 \leq 1/N$ for all $i = 1, ..., m$. Let $Q_1, ..., Q_r \in \mathrm{M}_n(\mathbb{C})$ be diagonal projections that form a partition of unity. For $X_1, ..., X_r$ a partition of $\{1, ..., n\}$ in such a way, that X_j consists of those elements $k \in \{1, ..., n\}$, where $(Q_j)_{kk} = 1$. Due to the choice of the vectors $v_1, ..., v_m$ we obtain that there exists $1 \leq j \leq r$ and $u \in \mathbb{C}^k$ with $||u|| = 1$ such that

$$\sum_{i \in X_j \cap \{1,...,m\}} |\langle u, v_i \rangle|^2 > N - 1 \;\Rightarrow\; \sum_{i \in X_j} |\langle u, w_i \rangle|^2 > 1 - \frac{1}{N} \tag{3.56}$$

So we end up with

$$\begin{aligned} ||Q_j P(\Phi u)||_2^2 \geq \sum \sum_{k=1}^{m} |\langle Q_j P(\Phi u), e_k \rangle|^2 &= \sum_{k \in X_j} |\langle \Phi u, e_k \rangle|^2 \\ &= \sum_{k \in X_j} |\langle u, w_k \rangle|^2 > 1 - \frac{1}{N} \end{aligned} \tag{3.57}$$

Consequently we have $||Q_j P Q_j|| = ||Q_j P||^2 > 1N^{-1}$. Consider the matrix $A = P - D$. Then, A has zero diagonal and satisfies $||A|| \leq 1 + N^{-1}$. By the previous discussion, it further follows that for any collection of $m \times m$ diagonal projections $Q_1, ..., Q_r$ with $\sum_k Q_k = \mathbb{1}$ we have

$$||Q_j A Q_j|| \geq ||Q_j P Q_j|| - ||Q_j D Q_j|| \geq 1 - \frac{2}{N} \tag{3.58}$$

Therefore we obtain for each N an example which falsify the Paving Conjecture 3.29. Hence the Paving Conjecture implies the KS_r conjecture. □

Chapter 4

The Resolution of the Problem

As we have seen in the last section, the original Kadison-Singer problem turned out to be equivalent to the Anderson-Paving conjecture as well as to Weaver's conjecture. However, the advantage of the formulation of Conjecture 3.32 is, that it reduces to finite dimensional vector spaces and hence to a question of linear algebra. The aim of this section is the presentation of the following probabilistic result due to Marcus, Spielman and Srivastava [27].

Theorem 4.1 (MSS). Let $\epsilon > 0$ and $v_1, ..., v_m \subset \mathbb{C}^d$ independent random vectors with finite support such that

$$\sum_{i=1}^{m} \mathbb{E}\left[v_i v_i^*\right] = \mathbb{1}_d \quad \text{and} \quad \mathbb{E}[||v_i||] \leq \epsilon \ \forall i \tag{4.1}$$

Then

$$\mathbb{P}\left[|| \sum_{i=1}^{m} v_i v_i^* || \leq (1+\sqrt{\epsilon})^2 \right] > 0 \tag{4.2}$$

Corollary 4.2. (MSS) implies Weaver's conjecture (KS_r)

Proof. Let $(u_i)_{i=1}^m \subset \mathbb{C}^d$ be a Parseval frame. We have to prove that Theorem 4.1 implies for each $r \in \mathbb{N}$ the existence of a partition $\{I_1, ..., I_r\}$ of $\{1, ..., m\}$ such that for each set of vectors $(u_i)_{i \in I_k}$ with $k = 1, ..., r$ we have

$$\sum_{i \in I_k} |\langle u, u_i \rangle|^2 \leq \left(\frac{1}{\sqrt{r}} + \sqrt{\delta} \right)^2 \tag{4.3}$$

To do so, let $r \in \mathbb{N}$ and let $v_1, ..., v_m$ be independent random vectors in $(\mathbb{C}^d)^{\oplus r} = \mathbb{C}^{rd}$, such that the vector v_i takes r different values given by $(\sqrt{r} u_i \otimes e_j)_{j=1}^r$, uniformly distributed. Then we have

$$\sum_{i=1}^{m} \mathbb{E}[v_i v_i^*] = \sum_{i=1}^{m} \left[\sum_{j=1}^{m} \sqrt{r}^2 u_j u_j^* \right] \otimes e_i e_i^* = \begin{bmatrix} \sum_{j=i}^m u_j u_j^* & & \\ & \ddots & \\ & & \sum_{j=i}^m u_j u_j^* \end{bmatrix} = \mathbb{1}_{dr} \tag{4.4}$$

by the Parseval frame property of the u_i. Further we have $\mathbb{E}[||v_i||^2] = \sum_{j=1}^r r^{-1}\sqrt{r}^2||u_i||^2 = r||u_i|| \leq \epsilon := r\delta$. Now we fulfill the requirements for $\epsilon = r\delta$ of Theorem 4.1. An application yields the existence of a realization such that the bound of (4.2) holds. For this realization define the partition of $\{1, ..., m\}$ as follows

$$I_k := \{i \in \{1, ..., m\} \,|\, v_i \text{ nonzero in } k^{\text{th}} \text{ component}\} \tag{4.5}$$

for $k = 1, ..., r$. By construction, for this particular realization $(v_i)_{i=1}^m$ we obtain the block diagonal matrix

$$\sum_{i=1}^m v_i v_i^* = \begin{bmatrix} r\sum_{j\in I_1} u_j u_j^* & & \\ & \ddots & \\ & & r\sum_{j\in I_r} u_j u_j^* \end{bmatrix} \tag{4.6}$$

and by assumption we have $||\sum_{i=1}^m v_i v_i^*|| \leq (1+\sqrt{\epsilon})^2$. With $\epsilon = r\delta$ this implies that the norm of each block is bounded by

$$||\sum_{i\in I_k} u_i u_i^*|| \leq \frac{1}{r}(1+\sqrt{\delta r})^2 = \left(\frac{1}{\sqrt{r}} + \sqrt{\delta}\right)^2 \tag{4.7}$$

what is exactly the bound from (4.3) and thus we obtain (KS_r). □

Before we are able to prove Theorem 4.1, we collect certain useful identities for matrices.

Lemma 4.3. The set $\mathrm{GL}(n, \mathbb{C})$ is a dense subset of $\mathrm{M}_n(\mathbb{C})$, i.e., $\overline{\mathrm{GL}(n, \mathbb{C})} = \mathrm{M}_n(\mathbb{C})$.

Proof. Let $A \in \mathrm{M}_n(\mathbb{C}) \setminus \mathrm{GL}(n, \mathbb{C})$ arbitrary. We know that an element of $\mathrm{M}_n(\mathbb{C})$ is not invertible if and only if it has an eigenvalue equal to zero. Let $\sigma(A)$ denote the spectrum of A and let $\lambda_1, ..., \lambda_r \in \mathbb{C}$ be the nonzero eigenvalues. Further let $\lambda_m := \min_i |\lambda_i|$. If $\epsilon > 0$ is given define $\delta := \min\{\frac{\epsilon}{n}, \frac{1}{2n}\lambda_m\}$. Then we have $\mathrm{GL}(n, \mathbb{C}) \ni B := A + \delta\mathbb{1}$ as well as $||A - B|| = \frac{\epsilon}{n}||\mathbb{1}|| \leq \epsilon$. □

We will now state Jacobi's formula, which expresses the derivative of the determinant of a matrix A in terms of the adjugate $\mathrm{adj}(A)$ of A and the derivative of A [52].

Theorem 4.4. Let $A : \mathbb{R} \to \mathrm{M}_d(\mathbb{C})$ be a differential map. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(A(t)) = \mathrm{tr}\left[\mathrm{adj}(A(t))\frac{\mathrm{d}A(t)}{\mathrm{d}t}\right] \tag{4.8}$$

Proof. We will first show that the differential of the the map $\det : \mathrm{M}_n(\mathbb{C}) \to \mathbb{C}$ evaluated at $\mathbb{1}$ coincides with the trace. For $A \in \mathrm{M}_n(\mathbb{C})$ we have

$$\det{}'(\mathbb{1})[A] = \nabla_A \det(\mathbb{1}) = \lim_{\epsilon\to 0}\frac{\det(\mathbb{1}+\epsilon A) - \det(\mathbb{1})}{\epsilon} \tag{4.9}$$

But $\det(\mathbb{1}+\epsilon A)$ is a polynomial in ϵ of degree n. In particular, it is related to the characteristic polynomial of A in the sense that the constant term is equal 1 and the linear term equal to $\mathrm{tr}(A)$. In the limit $\epsilon \to 0$, the only remaining term is $\mathrm{tr}(A)$. In the case that $A \in \mathrm{GL}(n, \mathbb{C})$

we also have the identity $\det'(A)[B] = \det(A)\det(A^{-1}B)$ for $B \in \mathrm{M}_n(\mathbb{C})$. To see this consider the function in X given by $\det(X) = \det(AA^{-1}X) = \det(A^{-1})\det(AX)$. Then we obtain

$$\det'(A)[B] = \det(A)\det'(\mathbb{1})[A^{-1}B] = \det(A)\det(A^{-1}B) \tag{4.10}$$

The claim now follows for the case $B = \partial_t A(t)$. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\det A = \det(A)\,\mathrm{tr}\left[A^{-1}\frac{\mathrm{d}A}{\mathrm{d}t}\right] = \mathrm{tr}\left[\mathrm{adj}(A)\frac{\mathrm{d}A}{\mathrm{d}t}\right] \tag{4.11}$$

By Theorem 4.3, the invertible matrices are dense in $M_n(\mathbb{C})$ and hence the identity must hold for all matrices. □

The next theorem insures the invariance of the number of zeros of an analytic function if it is subjected to small analytic deformations.

Theorem 4.5. Let f, g be holomorphic analytic functions defined on an elementary domain D and let γ be a closed curve in D which surrounds each point in its interior exactly ones. Further assume that f and $f+g$ only have finitely many zeros in D and that $|g(\xi)| < |f(\xi)|$ for all $\xi \in \mathrm{im}(\gamma)$. Then the functions f and $f+g$ have no zeros in the image of γ, and the functions f and $f+g$ have in the interior of γ the same number of zeros, counting multiplicities.

Proof. The family of functions

$$h_t(z) := f(z) + tg(z)\ ,\ \ t \in [0,1] \tag{4.12}$$

connects $f(z) = h_0(z)$ with $(f+g)(z) = h_1(z)$. Clearly, these functions have no zeros on the image of γ. The integral which counts the zeros in terms of the winding number

$$\frac{1}{2\pi i}\int_\gamma \frac{h_t'(\xi)}{h_t(\xi)}\,\mathrm{d}\xi \tag{4.13}$$

only yields natural numbers and is a continuous function in the parameter $t \in [0,1]$. But a continuous function with values in $\mathbb{N}$ must be constant. □

4.1 The mixed characteristic polynomial

Definition 4.6. A complex random variable Z on the probability space $(\Omega, \mathcal{A}, P)$ is a function $Z : \Omega \to \mathbb{C}$ such that both its real part $\Re(Z)$ and its imaginary part $\Im(Z)$ are real random variables on $(\Omega, \mathcal{A}, P)$. The expectation value of a complex random Z variable is defined as

$$\mathbb{E}[Z] = \mathbb{E}[\Re(Z)] + i\mathbb{E}[\Im(Z)] \tag{4.14}$$

Definition 4.7. Let $A_1, ..., A_m \in \mathrm{M}_n(\mathbb{C})$. For $z \in \mathbb{C}$ we call

$$\mu[A_1, ..., A_m](x) = \left(\prod_{i=1}^m (1 - \partial_{z_i})\right)\det\left(x\mathbb{1} + \sum_{i=1}^m z_i A_i\right)\Big|_{z_1=...=z_m=0} \tag{4.15}$$

the mixed characteristic polynomial.

Lemma 4.8. Let $A_1, ..., A_m \in \mathrm{M}_d(\mathbb{C})$. The mixed characteristic polynomial $\mu[A_1, ..., A_m](z)$ is a polynomial in $\mathbb{C}[z, z_1, ..., z_m]$ of degree $\leq d$. In particular, $\mu[A_1, ..., A_m](z)$ is a polynomial in $\mathbb{C}[z]$ of degree $\leq d$.

Proof. Define $B = z\mathbb{1} + \sum_{i=1}^{m} z_i A_i$ and apply Laplace expansion with respect to the jth row to det(B). This yields

$$\det(B) = \sum_{i=1} (-1)^{i+j} b_{ij} \det(B_{ij}) \tag{4.16}$$

where b_{ij} is the (i, j)th entry of B and B_{ij} is the $(d-1, d-1)$ submatrix which is obtained by eliminating the ith column and jth row. From this we obtain by an iterative process that det(B) is a polynomial in $\mathbb{C}[z, z_1, ..., z_m]$ of degree bounded by d. Hence $\mu[A_1, ..., A_m](z)$ is a polynomial in $\mathbb{C}[z]$ of degree no more then d. □

Mixed characteristic polynomials satisfy a number of interesting properties provided that $A_1, ..., A_m \geq 0$. In the following we will present certain properties that are necessary in order to prove Theorem 4.1.

Lemma 4.9. Let $z \in \mathbb{C}$ be arbitrary but fixed. Then the mixed characteristic polynomial mapping

$$\mu : \mathrm{M}_d(\mathbb{C}) \times \cdots \times \mathrm{M}_d(\mathbb{C}) \to \mathbb{C}\ ,\ (A_1, ..., A_m) \mapsto \mu[A_1, ..., A_m](x) \tag{4.17}$$

is multi-affine and symmetric, i.e., μ is an affine map in each variable and its value is invariant under the action of the symmetric group.

Proof. To prove the symmetry, let $\pi \in \mathfrak{S}_m$, where $\mathfrak{S}_m$ denotes the symmetric group on $\{1, ..., m\}$. We have

$$\det\left(\sum x\mathbb{1} + \sum_{i=1}^{m} z_{\pi(i)} A_{\pi(i)}\right)|_{z_{\pi(1)}=...=z_{\pi(m)}=0} = \det\left(x\mathbb{1} + \sum_{i=1}^{m} z_i A_i\right)_{z_1=...z_m=0} \tag{4.18}$$

By Lemma 4.8 we know that (4.18) is a polynomial and thus we can also conclude that we can interchange the order of the derivatives. This leads to $\mu[A_{\pi(1)}, ..., A_{\pi(m)}] = \mu[A_1, ...A_m]$. In order to show that μ is multi-affine it is sufficient to prove that for any $B \in \mathrm{M}_d(\mathbb{C})$ the function

$$f : \mathrm{M}_d(\mathbb{C}) \to \mathbb{C}\ ,\ A \mapsto f(A) := (1 - \partial_z) \det(B + zA)|_{z=0} \tag{4.19}$$

is affine. If $B \in \mathrm{GL}(d, \mathbb{C})$, Jacobi's formula (4.8) yields

$$f(A) = \det(B) - \det(B)\, \partial_z \det(\mathbb{1} + zB^{-1}A)|_{z=0} = \det(B)[1 - \mathrm{tr}(B^{-1}A)] \tag{4.20}$$

Since $\mathrm{GL}(d, \mathbb{C})$ is a dense open subset of $\mathrm{M}_d(\mathbb{C})$, we obtain the general case $B \in \mathrm{M}_d(\mathbb{C})$ by continuity. Hence, for any choice of $A_2, ..., A_m \in \mathrm{M}_d(\mathbb{C})$ the map

$$(\mathrm{M}_d(\mathbb{C}), \mathbb{C}^{m-1}) \ni (A, z_2, ..., z_m) \mapsto (1 - \partial_z) \det\left(z\mathbb{1} + zA + \sum_{i=2}^{m} z_i A_i\right)|_{z=0} \tag{4.21}$$

is an affine map in A and a polynomial of degree less then d in the variables $z_2, ..., z_m$. If one now applies linear operators, e.g., partial differential operators $(1 - \partial_{z_i})$, they will preserves this property. Thus the map $A \mapsto \mu[A, A_2, ..., A_m](z)$ is affine and by symmetry it is multi-affine. □

Lemma 4.10. Let $A_1, ..., A_m \in \mathrm{M}_d(\mathbb{C})$ such that $\mathrm{rk}(A_i) = 1$ for all $i = 1, ..., m$. Then the mixed characteristic polynomial of $A_1, ..., A_m$ is the characteristic polynomial of $A = A_1 + ... + A_m$. More precise, we have for $z \in \mathbb{C}$

$$\mu[A_1, ..., A_m](z) = \det[z\mathbb{1} - A] \tag{4.22}$$

Proof. If $A \in \mathrm{M}_d(\mathbb{C})$ is of rank one, there exists $u, v \in \mathbb{C}^d$ such that $A = uv^*$. Using the Sylvester determinant identity we obtain $\det(\mathbb{1} + tuv^*) = 1 + tv^*u$ for any $t \in \mathbb{C}$, what is an affine map. For any $A \in \mathrm{M}_d(\mathbb{C})$ consider the map $\mathbb{C} \ni t \mapsto \det(A + tuv^*)$. If $B \in \mathrm{GL}(d, \mathbb{C})$ we have

$$\det[B + tuv^*] = \det[B^{-1}(\mathbb{1} + tB^{-1}uv^*)] = \det[B]^{-1}\det[\mathbb{1} + t\tilde{u}v^*] \tag{4.23}$$

where we define $B^{-1}u =: \tilde{u}$. By the prior discussion, the map $t \mapsto \det(B + tuv^*) = b_0 + b_1 t$ with $b_0, b_1 \in \mathbb{C}$ is affine whenever B is invertible. Since $\mathrm{GL}(d, \mathbb{C})$ is a dense subset of $\mathrm{M}_d(\mathbb{C})$ we deduce the general case by continuity. If $z \in \mathbb{C}$ fixed this implies that the polynomial

$$p(z_1, ..., z_m) := \det\left[z\mathbb{1} + \sum_{i=1}^{m} z_i A_i\right] = c + \sum_{1 \leq i_1 < ... < i_j \leq m} a_{i_1} \cdot ... \cdot a_{i_j} z_{i_1} \cdot ... \cdot z_{i_j} \tag{4.24}$$

is affine multilinear in the complex variables $z_1, ..., z_m$. Evaluating p at the point $t \in \mathbb{C}^m$ can be written by using partial differential operators via

$$p(t_1, ..., t_m) = \left[\prod_{i=1}^{m}(1 + t_i \partial_{z_i})\right] p(z_1, ..., z_m)\big|_{z_1 = ... = z_m = 0} \tag{4.25}$$

Taking $t_i = -1$ for all $i = 1, ..., m$ yields □

$$\mu[A_1, ..., A_m](z) = p(-1, ..., -1) = \det[z\mathbb{1} - A] \tag{4.26}$$

Lemma 4.11. Let $X_1, ..., X_m$ be $d \times d$ jointly independent random matrices, which only take finitely many values. For $z \in \mathbb{C}$ we have

$$\mathbb{E}[\mu[X_1, ..., X_m](z)] = \mu[\mathbb{E}[X_1], ..., \mathbb{E}[X_m]](z) \tag{4.27}$$

Moreover, if $\mathrm{rk}(X_i) = 1$ for all $i = 1, ..., m$ we have

$$\mathbb{E}\left[\det\left(z\mathbb{1} - \sum_{i=1}^{m} X_i\right)\right] = \mu[\mathbb{E}[X_1], ..., \mathbb{E}[X_m]](z). \tag{4.28}$$

Proof. Let $B_1, ..., B_n, A_2, ..., A_m \in \mathrm{M}_d(\mathbb{C})$ and let $p_1, ..., p_n \in \mathbb{R}$ with $\sum_{i=1}^{n} p_i = 1$. By virtue of Lemma 4.9 we have for $z \in \mathbb{C}$

$$\mu\left[\sum_{i=1}^{n} p_i B_i, A_2, ..., A_m\right](z) = \sum_{i=1}^{n} p_i \mu[B_i, A_2, ..., A_m] \tag{4.29}$$

Since we assume $X_1, ..., X_m$ to be jointly independent, (4.27) follows. For the case that $\mathrm{rk}(X_i) = 1$ for all $i = 1, ..., m$, we can combine Lemma 4.10 together with (4.27) and obtain

$$\mu[\mathbb{E}[X_1], ..., \mathbb{E}[X_m]](z) = \mathbb{E}[\mu[X_1, ..., X_m](z)] = \mathbb{E}\left[\det\left(z\mathbb{1} - \sum_{i=1}^{m} X_i\right)\right] \tag{4.30}$$

what yields (4.28). □

4.2 Real stable polynomials

Real stable polynomials can be seen as a generalization of real-rooted polynomials. In the univariate case it is well known that complex roots of $p \in \mathbb{R}[z]$ come in conjugate pairs. Thus p is real rooted if and only if there are no roots with positive complex part.

Definition 4.12. A polynomial $f \in \mathbb{C}[z_1, ..., z_n]$ is called stable if $f(z_1, ..., z_n) \neq 0$ for all n-tuples $(z_1, ..., z_n) \in \mathbb{C}^n$ with $\Im(z_j) > 0$ for all $j = 1, ..., n$. If in addition f has real coefficients, it will be termed real stable. We will denote the set of stable and real stable polynomials by $\mathcal{H}_n(\mathbb{C})$ and $\mathcal{H}_n(\mathbb{R})$ respectively.

In the following we will frequently use the fact that the limit of real-rooted polynomials is again real-rooted. This can be seen as a consequence of the more general fact that the roots of a polynomial are continuous functions in its coefficients. When formalizing this statement, one also has to take the situation into account, where the sequence of polynomials may converge to a polynomial of lower degree, hence has strictly fewer roots. As an example consider the sequence

$$f_k(x) := \frac{1}{k}x^2 + x + 1 \tag{4.31}$$

where the roots of each f_k are given by $\frac{k}{2}(-1 \pm \sqrt{1 - 4n^{-1}})$.

Theorem 4.13 ([53]). Let $D \subset \mathbb{C}^n$ be a domain and suppose that $(f_k)_{k=1}^{\infty}$ is a sequence of nonvanishing analytic functions on D that converge to f uniformly on compact subsets of D. Then f is either nonvanishing on D or else identically zero.

Proof. We first prove the case $n = 1$ what can be seen as a consequence of the residue theorem. Assume the contrary, i.e., that $f \not\equiv 0$ but there exists a point $\zeta \in D \subset \mathbb{C}$ with $f(\zeta) = 0$. Then we can choose $\epsilon > 0$ in such a way, that the disk centered at ζ with radius 2ϵ is contained in the domain D and that there are no zeros of f in this disk except ζ. Consider the new sequence $g_n = f_n'/f_n$ which converges locally uniform in $U_{2\epsilon}(\zeta) \setminus \{\zeta\}$ to $g = f'/f$. Hence

$$0 = \frac{1}{2\pi i}\oint_{\partial U_\epsilon(\zeta)} \frac{f_n'}{f_n} \to \frac{1}{2\pi i}\oint_{\partial U_\epsilon(\zeta)} \frac{f'}{f} \tag{4.32}$$

what is a contradiction to the assumption $f(\zeta) = 0$. The case for multivariate functions can be obtained inductively from the univariate case. For this suppose that $f(\zeta) = 0$ for some $\zeta = (\zeta_1, ..., \zeta_n) \in D$. Further let $D' \subset D$ be a small polydisc centered at ζ. By the previously, we can apply Theorem 4.13 and can conclude that $f(z_1, \zeta_2, ..., \zeta_n) = 0$ for all z_1 such that $(z_1, \zeta_1, ..., \zeta_n) \in D'$. Applying the same argument repeatedly in the variables $z_2, ..., z_n$ we can conclude that f is identically zero on D' and thus, by analytic continuation, also on D. □

It is important to note that a real stable polynomial has real coefficients but may be evaluated on complex inputs. For the case $n = 1$, a real stable polynomial is a polynomial with real coefficients and real zeros. The question how rich the class of real stable polynomials is, is answered by the following

Lemma 4.14. Let $A_1, ..., A_n$ be positive semidefinite matrices. Then the polynomial

$$q(z_1, ..., z_n) = \det(z_1 A_1 + \cdots + z_n A_n) \tag{4.33}$$

is real stable.

Proof. By using Theorem 4.13, it is sufficient to prove the statement only for $A_j > 0$ where $j = 1, ..., n$. Define the function

$$z : \mathbb{C} \to \mathbb{C}^n \quad t \mapsto z(t) = \lambda t + \alpha \text{ with } \alpha \in \mathbb{R}^n \ , \ \lambda \in \mathbb{R}_+ \tag{4.34}$$

Further define the matrices $P := \sum_{i=1}^n \lambda_i A_i$, $H = B + \sum_{i=1}^n \alpha_i A_i$ and note that P is a positive definite matrix, hence there exist P^{-1} as well as $\sqrt{P}$. Then

$$f(z(t)) = \det \left[B + \sum_{i=1}^{n} (\alpha_i + \lambda_i t) A_i \right] = \det [tP + H] \tag{4.35}$$

$$= \det \left[\sqrt{P} (t\mathbb{1} + \sqrt{P}^{-1} H \sqrt{P}^{-1}) \sqrt{P} \right] = \det [P] \det \left[t\mathbb{1} + \sqrt{P}^{-1} H \sqrt{P}^{-1} \right] \tag{4.36}$$

Since the square root as well as the inverse of a positive definite matrix is still hermitian, (4.36) is just the product of some factor with the characteristic polynomial of a hermitian matrix and thus has only real eigenvalues. □

The statement of Lemma 4.14 is, that determinantal polynomials are real stable. However, for the bivariate case, by a theorem of Lewis-Parillo-Ramana [54], we also obtain that these are the only examples. That is, if $p(x, y)$ is real stable of degree d, then there exist real symmetric $d \times d$ positive semidefinite matrices A, B and C such that $p(x, y) = \pm \det [xA + yB + C]$. After introducing a new class of objects via imposing certain properties, it is a typical procedure in mathematics to investigate under which operations they are preserved. In our case, we are looking for operations which preserve real stability. Obviously, if $p \in \mathbb{R}[z_1, ..., z_n]$ is real stable, then $p(x, x, ..., x)$ is real-rooted.

Lemma 4.15. Let $p \in \mathbb{R}[z_1, ..., z_n]$ be real stable. The following operations preserve real stability

(1) restriction, i.e., for $t \in \mathbb{R}$ fixed, the polynomial $p(z_1, z_2, ..., z_n)|_{z_i = t}$ in $(n-1)$ variables is stable unless it is identically zero

(2) differentiation, i.e., if $t \in \mathbb{R}$, the polynomial $(1 + t\partial_{z_i})p$ is real stable for $i \in \{1, ..., n\}$

Proof. By virtue of Theorem 4.13 we know that if a sequence of non-vanishing holomorphic functions $(f_n)_{n \geq 1}$ on an open connected domain $\Omega \subset \mathbb{C}^m$ converges uniformly on compact sets, then its limit f is either non-vanishing or $f = 0$. If we set $\Omega := \mathbb{C}_+^{n-1}$, where $\mathbb{C}_+^n := \{z \in \mathbb{C}^n \,|\, \Im(z_i) > 0 \,\forall i\}$. For $p \in \mathbb{R}[z_1, ..., z_n]$ real stable, define the family of functions

$$f_k(z_1, ..., z_{n-1}) := p(t + \frac{i}{k}, z_1, ..., z_{n-1}) \text{ for } (z_1, ..., z_{n-1}) \in \Omega \tag{4.37}$$

We prove the differentiation property without loss of generality only for the variable z_1. Now $\lim_{k \to \infty} f_k(z_1, ..., z_{n-1}) = p(t, z_1, ..., z_{n-1})$ and by Theorem 4.13 this is a stable polynomial, thus yielding the restriction property. The differentiation property is clear for $t = 0$. Therefore assume $t \neq 0$ and fix $(z_2, ..., z_n) \in \Omega$. By definition, the polynomial $q(z) = p(z, z_2, ..., z_n) \in \mathbb{C}[z]$ is stable. Hence we can write $q(z) = c \prod_{i=1}^d (z - w_i)$ for some roots $w_1, ..., w_d \in \mathbb{C} \setminus \mathbb{C}_+^1$. Then we have

$$q(z) + tq'(z) = c \prod_{i=1}^{d} (z - w_i) \left(1 + \sum_{i=1}^{d} \frac{t}{z - w_i} \right) \tag{4.38}$$

Fix $z \in \mathbb{C}^1_+$. Since $\Im(w_i) \leq 0$, we have $z - w_i \in \mathbb{C}^1_+$ and consequently $\Im((z-w_i)^{-1}) < 0$ for all $i = 1, ..., d$. Hence, $\sum_{i=1}^d t(z-w_i)^{-1}$ has non-zero imaginary part. In particular we have $q(z) + tq'(z) \neq 0$ for any $z \in \mathbb{C}^1_+$. Since we assumed $(z_2, ..., z_n) \in \Omega$ to be arbitrary, we conclude that $(1 + t\partial_{z_1})p$ is real stable. □

Corollary 4.16. If $A_1, ..., A_n \in \mathrm{M}_d(\mathbb{C})$ are positive semidefinite and hermitian, then the mixed characteristic polynomial $\mu[A_1, ..., A_n]$ is real, stable and of degree d.

Proof. By definition of the mixed characteristic polynomial, we have

$$\mu[A_1, ..., A_n](z) = \left[\prod_{i=1}^n (1 - \partial_{z_i})\right] \det\left(z\mathbb{1} + \sum_{i=1}^n z_i A_i\right)|_{z_1=...=z_n=0} \tag{4.39}$$

Discarding the product of partial derivatives, Lemma 4.14 yields that this part is a real stable polynomial in $z \in \mathbb{C}$. The rest of the definition consists of taking partial derivatives and evaluating at certain points. Hence Lemma 4.15 yields that $\mu[A_1, ..., A_z](z)$ is real stable. □

Definition 4.17. Let $p \in \mathbb{R}[z]$ be real stable. We write maxroot(p) for the largest root of p.

Lemma 4.18. Let $p, q \in \mathbb{R}[z]$ be stable polynomials with $\deg(p) = \deg(q)$ and maxroot$(p) \leq$ maxroot(q). Suppose that every convex combination $(1-t)p + tq$ for $t \in [0,1]$ is also stable. Then for any $t_0 \in [0,1]$ we have

$$\text{maxroot}(p) \leq \text{maxroot}((1-t_0)p + t_0 q) \leq \text{maxroot}(q) \tag{4.40}$$

Proof. First define $m_q :=$ maxroot(q) and $m_p :=$ maxroot(p). Is $x > m_q$, both $p(x)$ and $q(x)$ are positive and hence $(1-t_0)p(x) + t_0 q(x) > 0$. Consequently, the polynomial $(1-t_0)p + t_0 q$ can not have any root larger than m_q what proves the second inequality of (4.40). We are now going to prove the first inequality by a contradiction. Suppose that $(1-t_0)p + t_0 q$ has no roots in $[m_p, m_q]$. This implies $(1-t_0)p + t_0 q > 0$ for all $x \geq m_p$. In particular, we have $q(m_p) > 0$. Thus, counting with multiplicity, q must have at least two roots to the right of m_p. Let D be an open disk in $\mathbb{C}$ centered at $\frac{1}{2}(m_p + m_q)$ with radius $\frac{1}{2}(m_q - m_p)$. We now show that

$$((1-t)p + tq)(z) \neq 0 \quad \forall z \in \partial D \ t_0 \leq t \leq 1 \tag{4.41}$$

Since the polynomial $(1-t)p + tq$ is stable by assumption, the claim directly follows for $z = m_p$ and $z = m_q$. The set $\partial D \times [0,1]$ is compact and hence

$$\inf_{(z,t) \in \partial D \times [t_0, 1]} |((1-t)p + tq)(z)| > 0 \tag{4.42}$$

By Theorem 4.5 the polynomials $(1-t)p + tq$ have the same numbers of zeros in D for all $t \in [t_0, 1]$. This yields a contradiction to the hypothesis that $(1-t_0)p + t_0 q$ has no roots in D, but q has at least two roots in D. □

4.3 Interlacing polynomials

One important ingredient in order to solve the Kadison-Singer Conjecture is the concept of interlacing for polynomials. As we will see, thus concept provides a way of reasoning about orderings of the roots of real-rooted polynomials. In particular, it allows us to infer information from the roots of the average polynomial about the roots of the individual polynomials.

Definition 4.19. Consider the polynomials

$$g(x) = \prod_{i=1}^{n-1} (x - \alpha_i) \quad \text{and} \quad f(x) = \prod_{i=1}^{n} (x - \beta_i) \tag{4.43}$$

We say that g interlaces f if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n \tag{4.44}$$

We say that polynomials $f_1, ..., f_k$ have a common interlacing if there is a single polynomial g that interlaces each of the f_i.

This definition can also be stated in an alternative way. Denote by β_{ij} the jth smallest root of the polynomial f_i. Then the polynomials $f_1, ..., f_k$ have a common interlacing if there exists an increasing sequence $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$ such that $\beta_{ij} \in [\alpha_{j-1}, \alpha_j]$ for all i and j. If the roots of the polynomial g are known, one can simply choose $\alpha_1, ..., \alpha_{n-1}$ to be the roots of g and α_0 respectively α_n can be chosen in such a way that it is smaller respectively larger than all of the roots of all the f_i.

Lemma 4.20. Let $f_1, ..., f_k$ be real rooted polynomials of degree n with positive leading coefficients and set

$$f_\emptyset = \sum_{i=1}^{k} f_i \tag{4.45}$$

If $f_1, ..., f_k$ have a common interlacing, then there exists an i for which the largest root of f_i is at most the largest root of $f_\emptyset$.

Proof. Let g be a polynomial that interlaces each of the f_i and suppose that α_{n-1} is its largest root. By assumption, each f_i has a positive leading coefficient and thus is positive for x sufficient large. As each f_i has exactly one root which is at least α_{n-1}, each f_i is nonpositive at α_{n-1}. Therefore the polynomial $f_\emptyset$ is as the sum of nonpositive polynomials nonpositive and eventually becomes positive. Hence $f_\emptyset$ has a root that is at least α_{n-1} and consequently its largest root is bounded from below by α_{n-1}. Let β_n be this root. It follows that there must be $i \in \{1, ..., n\}$ for which $f_i(\beta_n) \geq 0$. But f_i has at most one root that is at least α_{n-1} and $f_i(\alpha_{n-1}) \leq 0$. Thus the largest root of f_i is at least α_{n-1} and at most β_n. □

Definition 4.21. Let $I_1, ..., I_m$ finite index sets and for every assignment $i_1, ..., i_m \in I_1 \times \cdots \times I_m$, let $f_{i_1,...,i_m}(x)$ be a real rooted polynomial of degree n with positive leading coefficient. For $k < m$ we define the partial assignment $i_1, ..., i_k \in I_1 \times \cdots \times I_k$ via

$$f_{i_1,...,i_k}(x) = \sum_{i_{k+1},...,i_m} f_{i_1,...,i_k,i_{k+1},...,i_m} \tag{4.46}$$

where $i_{k+1}, ..., i_m \in I_{k+1} \times \cdots \times I_m$. Further, in analogy to Lemma 4.20 we define

$$f_\emptyset = \sum_{i_1,...,i_m} f_{i_1,...,i_m} \tag{4.47}$$

for $i_1, ..., i_m \in I_1 \times \cdots \times I_m$. We say that the polynomials $\{f_{i_1,...,i_m}\}_{i_1,...,i_m}$ form an interlacing family if for all $k = 0, ...m - 1$ and all $i_1, ..., i_k \in I_1 \times \cdots \times I_k$, the polynomials $\{f_{i_1,...,i_k,j}\}_{j \in I_{k+1}}$ have a common interlacing.

Theorem 4.22. Let $I_1, ..., I_m$ be finite index sets and let $\{f_{i_1,...,i_m}\}$ be an interlacing family of polynomials. Then there exists $i_1, ..., i_m \in I_1 \times \cdots \times I_m$ such that the largest root of $f_{i_1,...,i_m}$ is at most the largest root of $f_\emptyset$.

Proof. By definition of an interlacing family we know that the polynomials $\{f_t\}_{t\in S_1}$ have a common interlacing and that their sum is given by $f_\emptyset$. By virtue of Lemma 4.20 we can bound the largest root of one of the f_i by the largest root of $f_\emptyset$. The proof can now be completed via induction. For any $s_1, ..., s_k$ we know that the polynomials $\{f_{s_1,...,s_k,t}\}_{t\in S_{k+1}}$ have a common interlacing such that $\sum_{t\in S_{k+1}} f_{s_1,...,s_k,t} = f_{s_1,...,s_k}$. Therefore if we choose $t = s_{k+1}$, the largest root of the polynomial $f_{s_1,...,s_{k+1}}$ is at most the largest root of $f_{s_1,...,s_k}$. □

Lemma 4.23 ([55]). Let $f_1, ..., f_k$ be polynomials of the same degree with positive leading coefficients. Then $f_1, ..., f_k$ have a common interlacing if and only if $\sum_{i=1}^k \lambda_i f_i$ is real rooted for all $\lambda_1, ..., \lambda_k \geq 0$ such that $\sum_{i=1}^k \lambda_i = 1$.

4.4 Multivariate barrier argument

Definition 4.24. Let $p \in \mathbb{R}[z_1, ..., z_m]$. We say that $x \in \mathbb{R}^m$ is above the roots of the polynomial p if

$$p(x + t) > 0 \quad \forall\, t \in [0, \infty)^m \tag{4.48}$$

A barrier function of p in direction of the coordinate z_i, $i = 1, ..., m$ is defined for such x via

$$\Phi_p^i(x) = \partial_{z_i} \log[p(x)] = \frac{\partial_{z_i} p(x)}{p(x)} \tag{4.49}$$

In order to derive properties of the barrier function, we need an analog of the implicit function theorem for holomorphic functions.

Theorem 4.25 ([56]). Let $B \subset \mathbb{C}^n \times \mathbb{C}^m$ be an open set and $f = (f_1, ..., f_m) : B \to \mathbb{C}^m$ a holomorphic mapping. Further let $(z_0, w_0) \in B$ a point with $f(z_0, w_0) = 0$ and

$$\det\left[\left(\frac{\partial f_i}{\partial z_j}(z_0, w_0)\right)\right] \neq 0 \tag{4.50}$$

Then there exists an open neighbourhood $U := U_1 \times U_2 \subset B$ and a holomorphic map $g : U_1 \to U_2$ such that

$$\{(z, w) \in U_1 \times U_2 \,|\, f(z, w) = 0\rangle = \{(z, g(z)) \,|\, z \in U_1\} \tag{4.51}$$

Lemma 4.26. Let $p \in \mathbb{R}[z_1, z_2]$ be a stable polynomial. Then there exists a set $A \subset \mathbb{R}$ with $|A| < \infty$ such that for all $x \in \mathbb{R} \setminus A$ the polynomial $p(x, z_2) \in \mathbb{R}[z_2]$ has real roots and constant degree $d \in \mathbb{N}$. If $y_1(x) \leq \cdots \leq y_d(x)$ denote its roots counting multiplicity, then the map $x \mapsto y_i(x)$ is non-increasing for each $i \in \{1, ..., d\}$.

Proof. We can regard an element of $\mathbb{R}[z_1, z_2]$ as an element of $\mathbb{R}[z_1][z_2]$. More precisely this means that we can write $p(z_1, z_2) = \sum_{i=0}^d z_2^i q_i(z_1)$, where $q_i \in \mathbb{R}[z_1]$. Since p is real stable, Lemma 4.15 guarantees that for fixed $x \in \mathbb{R}$ the polynomial $p(x, z_2) \in \mathbb{R}[z_2]$ has only real roots. Clearly, $\deg(p(x, z_2)) = d$ if and only if $q_d(x) \neq 0$. In this case, the fundamental theorem of algebra implies the existence of roots $y_1(x) \leq \cdots \leq y_d(x)$. We now show that the map $x \mapsto y_i(x)$ is non-increasing for $i \in \{1, ..., d\}$. First we will verify that for every real root i.e, $(x, y) \in \mathbb{R}^2$ with $p(x, y) = 0$ we have $\partial_{z_1} p|_{(x,y)} \leq 0$ and $\partial_{z_2} p|_{(x,y)} \leq 0$. In order to generate a contradiction, suppose that $\alpha := \partial_{z_2} p(x, y) > 0$. Using Theorem 4.25, there exists open neighbourhoods $U_1, U_2 \subset \mathbb{C}$ of x, y respectively and a holomorphic function $g : U_1 \to U_2$ such that

$$\{(z_1, z_2) \in U_1 \times U_2 \,|\, p(z_1, z_2) = 0\} = \{(z, g(z)) \,|\, z \in U_1\} \tag{4.52}$$

If we now define $z_1 = x + \epsilon i$ and using that g is a holomorphic function, we have $g(z_1) \approx y + g'(x)\epsilon i = y + \alpha\epsilon i$ for small $\epsilon > 0$. But this produces a root of p whose imaginary part is non vanishing, which contradicts the stability of p. In the same manner we can treat p as a function in $\mathbb{R}[z_2][z_1]$ and thus we have $\partial_{z_1} p\,|_{(x,y)} \leq 0$ and $\partial_{z_2} p\,|_{(x,y)} \leq 0$. To finish the proof we will use the theory of algebraic curves, i.e., we will consider the set $\{(x, y) \in \mathbb{R} \,|p(x, y) = 0\}$. Every such algebraic curve decomposed as a finite union of smooth monotone arcs, which can be connected by some points, and a finite number of isolated points, called acnodes. In particular, a smooth monotone arc is the graph of some $y_i(x)$ restricted to some open interval. Differentiating $p(x, y_i(x)) = 0$ with respect to x yields

$$\partial_{z_1} p(z_1, z_2)\big|_{(x,y_i(x))} + \partial_{z_2} p(z_1, z_2)\big|_{(x,y_i(x))} \cdot y_i'(x) = 0 \tag{4.53}$$

But $\partial_{z_1} p\,|_{(x,y)} \leq 0$ and $\partial_{z_2} p\,|_{(x,y)} \leq 0$ and thus $y_i'(x) \leq 0$. □

Lemma 4.27. Let $p \in \mathbb{R}[z_1, ..., z_n]$ be stable and let $1 \leq i, j \leq n$. For any $k \in \mathbb{N}_0$ the partial derivatives of the barrier function of p satisfy

$$(-1)^k \frac{\partial^k}{\partial z_j^k} \Phi_p^i(x) \geq 0 \tag{4.54}$$

whenever $x \in \mathbb{R}^n$ is above the roots of p. More precise, the map $t \mapsto \Phi_p^i(x + te_j)$ is a non-negative, non-increasing and convex function of $t \geq 0$.

Proof. Let $p \in \mathbb{R}[z_1, ..., z_n]$ be real stable. We first consider the case $i = j$, i.e., we show that $(-1)^k \partial_{z_i}^k \Phi_p^i(x) \geq 0$ if $x \in \mathbb{R}^n$ is above the roots of p. If we fix all variables except z_i, then by Lemma 4.15 also this polynomial is real stable. Hence we can assume without loss of generality $p \in \mathbb{R}[z]$ real stable. Suppose that $x \in \mathbb{R}$ is above the roots of p. Recall that for a univariate polynomial stability is equivalent to the fact that p has only real roots. By the fundamental theorem of algebra there exist $y_1, ..., y_d, c \in \mathbb{R}$ such that $p(z) = c\prod_{i=1}^d (z - y_j)$. The barrier function of p is then given by

$$\Phi_p(x) = \frac{\partial_z p}{p}(x) = \sum_{i=1}^d \frac{1}{x - y_i} \;\Rightarrow\; (-1)^k (\partial_z^k \Phi_p)(x) = k! \sum_{i=1}^d \frac{1}{(x - y_i)^{k+1}} \tag{4.55}$$

By assumption x is above the roots of p and therefore $x > \max_i(y_i)$, hence all of the terms in (4.55) are positive. This proves the claim for the univariate case. The case $i \neq j$ and $k \geq 1$ remains. If $i \neq j$, we can fix all other variables z_k with $k \in \{1, ..., n\} \setminus \{i, j\}$ and by virtue of Lemma 4.15 obtaining still a real stable polynomial. By a relabeling of the variables, we can further assume $i = 1, j = 2$. Assume that $\mathbb{R}^2 \ni x = (x_1, x_2)$ is above the roots of $p \in \mathbb{R}[z_1, z_2]$ stable. By the Schwartz-theorem we obtain

$$(-1)^k \partial_{z_2}^k \Phi_p^1(x) = ((-1)^k \partial_{z_2}^k \partial_{z_1} \log[p])(x) = \partial_{z_1}((-1)^k \partial_{z_2}^k \log(p))(x) \tag{4.56}$$

it is sufficient to show that the map $x_1 \mapsto ((-1)^k \partial_{z_2}^k \log[p])(x_1, x_2)$ is non-decreasing for $x_2 \in \mathbb{R}$ fixed. As we have seen in the proof of Lemma 4.26, we can write $p(x_1, x_2) = c(x_1) \prod_{i=1}^d (x_2 - y_i(x_1))$. This implies

$$((-1)^k \partial_{z_2}^k \log[p])(x_1, x_2) = -(k-1)! \sum_{i=1}^d \frac{1}{(x_2 - y_i(x_1))^k} \tag{4.57}$$

Since x is by assumption above the roots of p, we have $x_2 > \max_i(y_i(x))$ and it follows from Lemma 4.26 that the above function is non-decreasing. □

Lemma 4.28. Let $p \in \mathbb{R}[z_1, ..., z_n]$ be stable. Suppose that $x \in \mathbb{R}$ lies above the roots of p and that for some $j \in \{1, ..., n\}$ and $\delta > 0$ we have

$$\Phi_p^j(x) \leq 1 - \frac{1}{\delta} \tag{4.58}$$

Then $x + \delta e_j$ lies above the roots of $(1 - \partial_{z_j})p$ and for all $i = 1, ..., n$ we have

$$\Phi^i_{(1-\partial_{z_j})p}(x + \delta e_j) \leq \Phi_p^i(x) \tag{4.59}$$

Proof. Let $y \in \mathbb{R}^n$ such that y lies above x, that is, $y_i \geq x_i$ for every $i = 1, ..., n$. Using the monotonicity statement from Lemma 4.27 we have $\Phi_p^i(y) \leq \Phi_p^i(x) < 1$ for all $i = 1, ..., n$, where the second bound is a consequence from the assumption. Therefore $1 - \Phi_p^i(y) > 0$ and thus

$$((1 - \partial_{z_i})p(z_1, ..., z_m))|_y = p(y)(1 - \Phi_p^i(y)) > 0 \tag{4.60}$$

In particular, $\mathbb{R}^n \ni x + \delta e_i$ is above the roots of $(1 - \partial_{z_i})p$. Since (4.60) guarantees that $(1 - \partial_{z_i})p(y) > 0$, we can apply log and obtain

$$\log[(1 - \partial_{z_i})q](y) = \log[p(y)(1 - \Phi_p^i(y))] = \log[p(y)] + \log[(1 - \Phi_p^i)(y)] \tag{4.61}$$

Further, an application of ∂_{z_k} for $k = 1, ..., n$ to (4.61) yields

$$\Phi^k_{(1-\partial_{z_i})p}(y) = \Phi_p^k(y) - \frac{\partial_{z_k} \Phi_p^i(y)}{1 - \Phi_p^i(y)} \tag{4.62}$$

Since $\partial_{z_k} \Phi_p^i(y) = \partial_{z_k} \partial_{z_i} \log[p(y)] = \partial_{z_i} \partial_{z_k} \log[p(y)] = \partial_{z_i} \Phi_p^k(y)$ the required bound is equivalent to

$$\frac{\partial_{z_j} \Phi_p^k(x + \delta e_i)}{1 - \Phi_p^i(x + \delta e_i)} = \Phi_p^k(x + \delta e_i) - \Phi^i_{(1-\partial_{z_i})p}(x + \delta e_i) \geq \Phi_p^k(x + \delta e_i) - \Phi_p^k(x) \tag{4.63}$$

Further, using Lemma 4.27 we have $\Phi_p^k(x + \delta e_i) - \Phi_p^k(x) \leq \delta\, \partial_{z_i} \Phi_p^j(x \delta e_i) \leq 0$. Consequently, (4.63) is implied by multiplying the inequality

$$\frac{1}{1 - \Phi_p^i(x + \delta e_i)} \leq \delta \tag{4.64}$$

with $\partial_{z_i} \Phi_p^k(x + \delta e_i)$. In addition, (4.64) is a consequence of Lemma 4.27 and the assumption (4.58), since

$$\Phi_p^i(x + \delta e_i) \leq \Phi_p^i(x) \leq 1 - \frac{1}{\delta} \tag{4.65}$$

□

Corollary 4.29. Let $p \in \mathbb{R}[z_1, ..., z_m]$ be stable and suppose that $x \in \mathbb{R}^m$ lies above the roots of p and that there exists $\delta > 0$ such that for $j = 1, ..., m$

$$\Phi_p^j(x) \leq 1 - \frac{1}{\delta} \tag{4.66}$$

holds. Then $(x + (\delta, ..., \delta)) \in \mathbb{R}^m$ lies above the roots of $\Pi_{i=1}^m (1 - \partial_{z_i})p$.

Proof. Define $n + 1$ new vectors and polynomials for $i \in \{0, ..., n\}$ via

$$y_i := x + \delta \sum_{j=1}^{n} e_j \quad \text{and} \quad p_i := \prod_{k=1}^{i} (1 - \partial_{z_k})p \tag{4.67}$$

If we now apply Lemma 4.28 iterative, we obtain that y_i lies above of p_i for all $i \in \{1, ..., n\}$. □

4.5 Synthese

Theorem 4.30 (MCP). Let $\epsilon > 0$ arbitrary and suppose that $A_1, ..., A_n \in \mathrm{M}_d(\mathbb{C})$ positive semidefinite such that

$$\sum_{i=1}^{n} A_i = \mathbb{1} \quad \text{and} \quad \mathrm{tr}[A_i] \leq \epsilon \ \forall i \tag{4.68}$$

Then all roots of the mixed characteristic polynomial $\mu[A_1, ..., A_n]$ are real and the largest root is bounded by $(1 + \sqrt{\epsilon})^2$.

Proof. Given $A_1, ..., A_n \in \mathrm{M}_d(\mathbb{C})$ define the polynomial $p \in \mathbb{R}[z_1, ..., z_n]$ via

$$p(z_1, ..., z_n) = \det \left[\sum_{i=1}^{n} z_i A_i \right] \tag{4.69}$$

From Lemma 4.14 and Lemma 4.15 we know that p is a real stable polynomial. As in the proof of Lemma 4.14 an application of Jacobi's formula (4.8) yields for any $i \in \{1, ..., n\}$ and $x \in \mathbb{R}^n$

$$\partial_{z_i} p = \partial_t \det \left(\sum_{k=1}^{n} x_k A_k + t A_i \right) \Big|_{t=0} = \det \left(\sum_{k=1}^{n} x_k A_k \right) \mathrm{tr} \left[\left(\sum_{k=1}^{n} x_k A_k \right)^{-1} A_i \right] \tag{4.70}$$

Using the assumptions given in (4.68) we obtain for $t > 0$

$$\begin{aligned} \Phi_p^i(t, ..., t) &= \frac{\partial_{z_i} p}{p}((t, ..., t)) = \frac{\det \left[\sum_{k=1}^{n} t A_k \right] \mathrm{tr} \left[\left(\sum_{k=1}^{n} t A_k \right)^{-1} A_i \right]}{\det \left[\sum_{k=1}^{n} t A_k \right]} \\ &= \mathrm{tr} \left[\left(\sum_{k=1}^{n} t A_k \right)^{-1} A_i \right] = \mathrm{tr}[t^{-1} \mathbb{1}^{-1} A_i] = t^{-1} \mathrm{tr}[A_i] \leq t^{-1} \epsilon \end{aligned} \tag{4.71}$$

Further $x = (t, ..., t)$ lies above the roots of p for any $t > 0$. In order to make use of Corollary 4.29 we need

$$\frac{\epsilon}{t} \leq 1 - \frac{1}{\delta} \Leftrightarrow \frac{\epsilon}{t} + \frac{1}{\delta} \leq 1 \tag{4.72}$$

Choose $t, \delta > 0$ accordingly. Then by Corollary 4.29 $(t+\delta, ..., t+\delta)$ lies above the roots of $\Pi_{i=1}^{n}(1-\partial_{z_i})p$. Since

$$\prod_{i=1}^{n}(1-\partial_{z_i})p(z_1, ..., z_n)\big|_{z_1=...=z_n=z} = \mu[A_1, ..., A_n](z) \tag{4.73}$$

for any $z \in \mathbb{C}$ we can conclude that $\operatorname{maxroot}[\mu[A_1, ..., A_n]] \leq t + \delta$. If we now minimize $t+\delta$ under the constraint given in (4.72) one obtains $t_{\text{min}} = \sqrt{\epsilon} + \epsilon$ and $\delta_{\text{min}} = 1 + \sqrt{\epsilon}$. Hence the best bound for the largest root is $t_{\text{min}} + \delta_{\text{min}} = 1 + 2\sqrt{\epsilon} + \epsilon = (1+\sqrt{\epsilon})^2$. □

Finally we are able to prove Theorem 4.1 which will also make the importance of Theorem 4.30 evident.

Theorem 4.31. Theorem 4.30 implies Theorem 4.1.

Proof. Let $v_1, ..., v_n \subset \mathbb{C}^d$ be random vectors as in (MSS) and consider the rank one positive semidefinite random matrices $X_i := v_i v_i^*$ for $i = 1, ..., n$. Since the X_i are assumed to be hermitian, an application of Lemma 4.10 yields

$$||\sum_{i=1}^{n} v_i v_i^*|| = ||\sum_{i=1}^{n} X_i|| = \operatorname{maxroot}\left[\det\left(z\mathbb{1} - \sum_{i=1}^{n} X_i\right)\right] = \operatorname{maxroot}[\mu[X_1, ..., X_n]] \tag{4.74}$$

By Lemma 4.20 the bound on $\operatorname{maxroot}[\mu[A_1, ..., A_n]]$ with $A_i := \mathbb{E}[X_i]$ in the conclusion of Theorem 4.30 yields the same bound on $||\sum_{i=1}^{n} v_i v_i^*||$ with positive probability. □

Chapter 5

The Relatives of the Problem

In this section we will explore two more problems that are closely related to the Kadison-Singer problem and its proof. In particular, the existence of an infinite sequence of Ramanujan graphs of degree d was proven by Marcus, Spielman and Srivastava using the machinery of interlacing polynomials before they applied it to Weaver's conjecture. Since here the argumentation does not involve the multivariate barrier argument, and solely relies on real stability and the interlacing properties, the logic of the argument appears much clearer. Finally, we return to equivalent formulations of the Kadison-Singer problem in the field of linear algebra. Here we show its equivalence to the Bourgain-Tzafriri conjecture, which is closely related to the restricted invertibility theorem.

5.1 Ramanujan graphs

The structure of this section is as follows. We will first revisit important concepts of graph theory with a focus on the adjacency matrix of a graph and properties of its eigenvalues. Most important, the Alon-Boppana Theorem 5.6 points out the optimality of Ramanujan graphs. We are then going to describe a method that allows for the construction of a graph from a given graph with twice the vertices. This construction is not unique and some additional data is needed, namely the so called signing. One of the key observations is, that the family of all those graphs form an interlacing family, which expectation value equals the so called matching polynomial. However, for this polynomial certain bounds on its largest root are known. In fact, the roots of this polynomial can be bounded by $2\sqrt{d-1}$, as desired. Using the properties of interlacing families derived in Section 4, we can infer that there must exist a graph signing, whose roots are all bounded by $2\sqrt{d-1}$. This provides a method for the construction of infinite families of Ramanujan graphs of any degree d.

Definition 5.1. A graph G is a pair $G = (V, E)$ consisting of a vertex set $V = V(G)$ and a edge set $E = E(G)$ which is a set of two-sets of vertices. The vertices u and v of an edge $e = \{u, v\}$ are called the endpoints of an edge. A loop is an edge whose endpoints are equal. When two vertices u and v are endpoints of an edge, we say that they are adjacent and write $u \sim v$ to indicate this. A stable set is a set of vertices in a graph, no two of which are adjacent. A graph G is called bipartite, if its vertices can be divided into two disjoint and stable sets J, K such that every edge connects a vertex in J to one in K. In this case, the vertex sets J, K are called the parts of the graph.

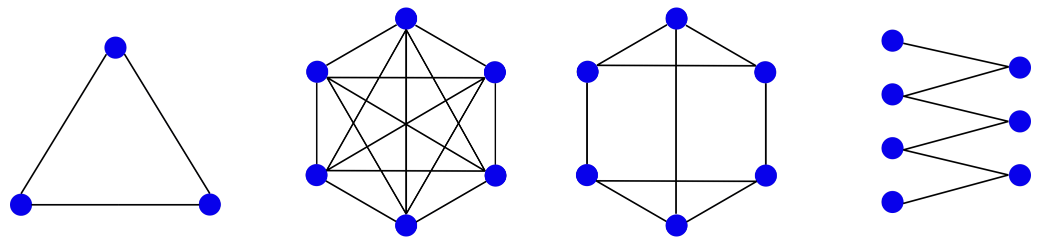


Figure 5.1: Examples of graphs. The first graph is the complete graph K_3 (a graph where for each pair of vertices there exists an edge connecting them) on three vertices and has 3 edges. The second is the complete graph on six vertices K_6 and has 15 edges. In general, a complete graph on n vertices K_n has $\frac{1}{2}n(n-1)$ edges. The third graph is a 3-regular graph with 6 vertices, i.e., each vertex has the same number of neighbours. The fourth graph is a bipartite graph with 7 vertices, where the first stable set contains 4 vertices and the second 3 vertices.

To any graph G one can associate the adjacency matrix $A \in \mathrm{M}_n(\mathbb{R})$ where $n = |V|$. The rows and columns are indexed by the elements of the vertex set and the (u, v)-th entry is the number of edges connecting u and v. Since our definition of a graph does not comprise any information about the direction of the edges, the matrix A is symmetric, i.e., $A^\top = A$ and consequently $\sigma(A) \subset \mathbb{R}$. The degree $\deg(v)$ of a vertex v is the number of edges incident with v, where a loop is counted with multiplicity 2. With this convention it is easy to see that

$$\sum_{v \in V} \deg(v) = 2|E(G)| \tag{5.1}$$

A graph is said to be connected if for any $u, v \in V$ there is a path from u to v. A graph is said to be k-regular if every vertex has degree k.

Theorem 5.2. Let G be a graph and A its adjacency matrix. Further denote by $\Delta(G)$ the maximal degree of any vertex $v \in V$. If λ is an eigenvalue of A then $|\lambda| \leq \Delta(G)$.

Proof. If $x = (x_1, ..., x_n)^\top \in \mathbb{R}^n$ is an eigenvector of A with eigenvalue λ, by definition we have $\lambda x = Ax$. Without loss of generality we can assume that $|x_1| = \max_{1 \leq i \leq n} |x_i|$. Then

$$|\lambda|\,|x_1| = |\sum_{j=1}^{n} a_{1j} x_j| \leq |x_1| \sum_{j=1}^{n} a_{1j} = |x_1| \deg(v_1) \leq |x_1| \Delta(G) \tag{5.2}$$

Since $x \neq 0$ we have $x_1 > 0$ and therefore we obtain $|\lambda| \leq \Delta(G)$ □

Corollary 5.3. If G is a k-regular graph, then all the eigenvalues λ of its adjacency matrix satisfy $|\lambda| \leq k$.

Obviously a k-regular graph is one whose adjacency matrix has every row sum and hence every column sum equal to k. Thus $\lambda = k$ is an eigenvalue of A with corresponding eigenvector $u = (1, ..., 1)^\top$.

Theorem 5.4. If G is a k-regular graph, then $\lambda = k$ is an eigenvalue with multiplicity equal to the number of connected components of G.

Proof. We have already argued that $\lambda = k$ is an eigenvalue of a k-regular graph. Suppose that $v = (v_1, ..., v_n)$ is an eigenvector of the adjacency matrix A of G with eigenvalue k. Further we can assume without loss of generality that $|v_1| = \max_i |v_i|$ as well as $x_1 > 0$. It follows that

$$kv_1 = \sum_{j=1}^{n} a_{1j}v_j \le \sum_{j=1}^{n} a_{1j}v_1 = kv_1 \tag{5.3}$$

This means, that we have $v_j = v_1$ for every j for which $a_{1j} \neq 0$. More precise, this holds for all those j, for which u_j is adjacent with u_1. Iterating the argument with each of the vertices in the neighbourhood of u_1, we obtain that $v_j = v_1$ if u_j is connected to v_1. Applying this argument to each component of the graph, the claim follows. □

By virtue of Theorem 5.4 we can order the eigenvalues of a connected k-regular graph G in the following chain

$$k = \lambda_0 (G) > \lambda_1 (G) \ge \cdots \ge \lambda_{n-1} (G) \ge -k \tag{5.4}$$

If the graph G is bipartite and k-regular, one can divide the vertex set in two disjoint and stable sets J and K. Label the elements of J by numbers $1, ..., |J|$ and the elements of K with $|J| + 1, ..., |G|$. Since the graph is regular, each element of G has degree k. In particular, every element of J ,K has degree k and since G is also bipartite, we have $|J| = |K|$. Therefore the adjacency matrix is of the form

$$A = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \quad \text{where} \quad \sum_{j=1}^{|J|} B_{ij} = \sum_{i=1}^{|K|} B_{ij} = k \tag{5.5}$$

It can be easily seen that the vector $x = (1, ..., 1, -1, ..., -1)$ is an eigenvector of A to the eigenvalue $-k$. To sum up, the adjacency matrix A of a k-regular graph has eigenvalues k and $-k$ if and only if G is bipartite. The eigenvalues k, and $-k$ if G is bipartite, are called the trivial eigenvalues of A.

Definition 5.5. Let $G = (V, E)$ with $|V| = n$ be a k-regular graph and denote by $\lambda_{\text{max}} (G)$ the absolute value of the largest nontrivial eigenvalue of its adjacency matrix. We call G a Ramanujan graph if

$$\lambda_{\text{max}} (G) \le 2\sqrt{k-1} \tag{5.6}$$

If the number of vertices is small, it is easy to construct Ramanujan graphs e.g., k-regular complete graphs and complete bipartite graphs are Ramanujan graphs. The construction of such graphs becomes more involved if one wants to obtain an infinite family of k-regular graphs that are all Ramanujan. The importance of the number $2\sqrt{k-1}$ in Definition 5.5 is due to the following

Theorem 5.6 (Alon-Boppana, [30]). Let $G = (V, E)$ be a d-regular graph and Δ its diameter. Then we have

$$\lambda_2 \ge 2\sqrt{d-1} - \mathcal{O}\left(\frac{\sqrt{d-1}}{\Delta - 1}\right) \tag{5.7}$$

Proof. The idea of the proof is to explicitly construct an eigenvector of the adjacency matrix A of G, whose eigenvalue has the desired property. Let $a, b \in V$ such that $d(a,b) = \Delta(G)$ and define $\kappa := \lfloor \frac{\Delta(G)-4}{2} \rfloor$. Further let $\tilde{a}$ be a neighbour of a. For $v \in V$ we call

$$d(v, \{a, \tilde{a}\}) := \min \{d(v,a), d(v,\tilde{a})\} \tag{5.8}$$

Consider a vector $x \in \mathbb{R}^{|V|}$ given by

$$\begin{cases} x_v = \frac{1}{\sqrt{d-1}^{\alpha}} & \text{if } d(v,\{a,\tilde{a}\}) =: \alpha \leq \kappa \\ x_v = 0 & \text{if } d(v,\{a,\tilde{a}\}) > \kappa \end{cases} \tag{5.9}$$

We are going to show that

$$x^\top A x \geq 2\sqrt{d-1}||x||^2 \left[1 - \frac{1}{\kappa - 1}\right] \tag{5.10}$$

In order to achieve this, let us formulate the statement by using the Laplacian matrix of G, i.e., $L = d\,\mathbb{1} - A$. For the quadratic form of L, we have the following expression

$$x^\top L x = d x^\top x - x^\top A x = \sum_{(u,v)\in E} (x_u - x_v)^2 \tag{5.11}$$

For a given vertex $v \in V$ denote by S_v the set of neighbours $u \in V$ of v such that $x_u < x_v$. By the d-regularity of the graph we clearly have $|S_v| \leq d-1$. Further define L_α as the set of those vertices at distance exactly α to the set $\{a, \tilde{a}\}$. We then have

$$\begin{aligned}
\sum_{(u,v)\in E} (x_u - x_v)^2 &= \sum_{\alpha\leq\kappa} \sum_{u\in L_\alpha} \sum_{v\in S_u} (x_u - x_v)^2 \\
&= \sum_{\alpha\leq\kappa-1} \sum_{u\in L_\alpha} \sum_{v\in S_u} (x_u - x_v)^2 + \sum_{u\in L_\kappa} \sum_{v\in S_u} x_u^2 \\
&= \sum_{\alpha\leq\kappa-1} \sum_{u\in L_\alpha} |S_u| \left[x_u - \frac{x_u}{\sqrt{d-1}}\right]^2 + \sum_{u\in L_\kappa} |S_u| x_u^2 \\
&\leq \sum_{\alpha\leq\kappa-1} \sum_{u\in L_\alpha} (d-1) \left[x_u - \frac{x_u}{\sqrt{d-1}}\right]^2 + \sum_{u\in L_\kappa} (d-1) x_u^2 \qquad (5.12) \\
&= \sum_{\alpha\leq\kappa-1} \sum_{u\in L_\alpha} x_u^2 (\sqrt{d-1} - 1)^2 + \sum_{u\in L_\kappa} (d-1) x_u^2 \\
&= \sum_{u\in L_\kappa} (d-1) x_u^2 + \sum_{\alpha\leq\kappa-1} \sum_{u\in L_\alpha} x_u^2 (d - 2\sqrt{d-1}) \\
&= \sum_{u\in V} d x_u^2 - \sum_{u\in V} 2\sqrt{d-1} x_u^2 + \sum_{u\in L_\kappa} (2\sqrt{d-1} - 1) x_u^2
\end{aligned}$$

In total, we arrive at

$$\sum_{u\in L_\kappa} x_u^2 = |L_\kappa| \frac{1}{(d-1)^\kappa} \leq \frac{1}{\kappa+1} \sum_{\alpha=0}^{\kappa} |L_\alpha| \frac{1}{(d-1)^\alpha} = \frac{1}{\kappa+1} \sum_{u\in V} x_u^2 \tag{5.13}$$

since the function $\alpha \mapsto |L_\alpha| (d-1)^{-\alpha}$ is decreasing for increasing α. For the quadratic form induced by the Laplacian this implies

$$x^\top L x \leq d\,||x||^2 - 2\sqrt{d-1}\,||x||^2 + \frac{2\sqrt{d-1} - 1}{\kappa + 1} ||x||^2 \tag{5.14}$$

and for the quadratic form of the adjacency matrix

$$x^\top A x \geq \left[2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{\kappa+1}\right] ||x||^2 \tag{5.15}$$

Let $y \in \mathbb{R}^{|V|}$ be a vector constructed according (5.9) but now for some vertex $b \in V$ and $\tilde{b} \in V$ a neighbour of b. Clearly, it is not possible for a vertex at distance $\leq \kappa$ to $\{a, \tilde{a}\}$ to also be at distance $\leq \kappa$ to $\{b, \tilde{b}\}$, since otherwise we would have a path of length $\leq 2\kappa + 2$ from a to b. Hence, the vectors x and y are non-zero on disjoint subsets of coordinate, and hence are orthogonal. In particular, if we consider any linear combination $\xi = \beta_1 x + \beta_2 y$ we obtain

$$\xi^\top A \xi = \beta_1^2 x^\top A x + \beta_2^2 y^\top A y \geq \left[2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{\kappa+1}\right] (\beta_1^2 ||x||^2 + \beta_2^2 ||y||^2) \tag{5.16}$$

Further we have by the orthogonality of x, y, that $\beta_1^2 ||x||^2 + \beta_2^2 ||y||^2 = ||\xi||^2$. Therefore we have constructed a two-dimensional set of vectors whose quotient is at most the above expression, and thus

$$\lambda_2 \geq 2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{\kappa+1} \tag{5.17}$$

□

With other words, Theorem 5.6 tells us, that for any $\epsilon > 0$ there exists an $n \in \mathbb{N}$, such that any d-regular graph with n vertices has a nontrivial eigenvalue, which absolute value is at least $2\sqrt{d-1} - \epsilon$. Therefore, if one wants to construct graphs where the nontrivial eigenvalues are as small as possible, $2\sqrt{k-1}$ serves as the lower limit of what can be done. With respect to this property, Ramanujan graphs are optimal. Note, that one can sharpen the bound on λ_2 given in (5.7). More precise, Fiedman [31] showed that for any d-regular graph G with $|V(G)| = n$, one has $\lambda_2 \geq 2\sqrt{d-1}(1 - \mathcal{O}(\log(n)^{-2}))$.

Further important ingredients include so called covering maps, lifts, the universal covering tree as well as the path tree. The idea behind all four concepts is to view the graph from a topological perspective, that is, as a one-dimensional simplicial complex. For a complete review see [57, 58]. In particular, lifts will turn out to be a useful tool in order to produce new larger graphs from a given one and the universal covering tree will bound the largest eigenvalue of the matching polynomial.

Definition 5.7. Let G and H be two graphs. We say that a function $f : V(H) \to V(G)$ is a covering map if for every $v \in V(H)$, f maps the neighbourhood set $N_H(v) \subset V(H)$ of a vertex v bijective onto $N_G(f(v))$. If there exists a covering function from H to G, we say that H is a lift of G or that G is a quotient of H. If f is a covering map onto G and $v \in V(G)$, we call the set $f^{-1}(v)$ the fibre of v. Similarly, if $e \in E(G)$, we say that $f^{-1}(e)$ is the fibre of e. If $|f^{-1}(v)| = |f^{-1}(e)| = n$ for all $v \in V(G)$ and all $e \in E(G)$, we call n the covering number.

Example 5.8. While Definition 5.7 treats lifts in a very general way, we are in the following mainly interested in 2-lifts, which have a very pleasing description. Let G be a graph and denote by A the adjacency matrix of G. A signing $\tilde{A}$ of A is a symmetric matrix that is obtained by replacing some of the 1-entries in A by -1. In fact, there exists a bijective correspondence between 2-lifts of G and signings of A. One assigns to each edge $e \in E(G)$

a value from $\{\pm1\}$, what yields a sequence $s \in \{\pm1\}^{|E(G)|}$. Then, one constructs a graph $\tilde{G}$ where for each vertex $v \in V(G)$ there are two vertices $\{v_0, v_1\} \subset V(\tilde{G})$ in $\tilde{G}$ and for each edge $(u, v) \in E$ there are two edges in $E(\tilde{G})$, given by one of the following options

(1) $\{(u_0, v_0), (u_1, v_1)\}$ if the signing was $+1$

(2) $\{(u_0, v_1), (u_1, v_0)\}$ if the signing was -1

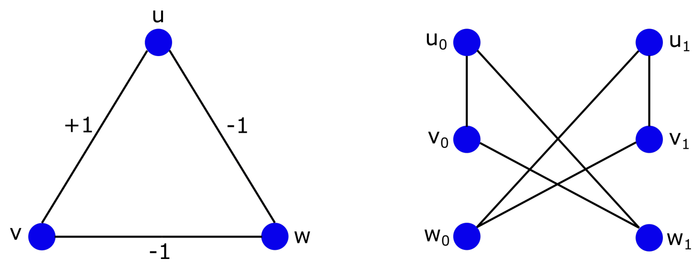


Figure 5.2: A 2-lift of the 2-regular graph on three vertices. First, we assign to each edge a number $\{\pm1\}$. Then we double the number of vertices, where $\{u_0, u_1\}$ is the fibre of the original vertex u. Accordingly to the assignment, we have $s(u, v) = 1$, hence the edges in the 2lift are of form (1) and we have to connect (u_0, v_0) and (u_1, v_1).

We have already seen in Figure 5.1 certain graphs with a small number of vertices that turn out to be all Ramanujan. This emphasizes the fact, that if the number of vertices is small, the fraction of those graphs which are Ramanujan is high. In particular, Theorem 5.6 implies that the problem becomes hard for large graphs. A partial result regarding the existence of Ramanujan graphs for a large number of vertices was proven in 1988 by Lubotzky, Phillips and Sarnak [29], Margulis [59] and 1994 by Morgenstern [60].

Theorem 5.9 ([57]). For every prime p and every positive integer k, there exist infinitely many d-regular Ramanujan graphs with $d = p^k + 1$.

Using the machinery developed in Section 4, we are able to prove the following result, which generalizes Theorem 5.9 and gives an affirmative answer to the question of existence of large k-regular Ramanujan graphs for arbitrary k.

Theorem 5.10 ([28]). For every $d \geq 3$ there exists an infinite sequence of d-regular bipartite Ramanujan graphs.

In order to prove Theorem 5.10, we will not investigate the eigenvalues of the adjacency matrix directly, but introduce an auxiliary quantity, namely the so called matching polynomial μ_G. In fact it turns out (cf. Theorem 5.18), that the matching polynomial does not contain the whole information about the eigenvalues of one particular graph, but about the average over all signings of that graphs.

Definition 5.11. A set M of independent edges in a graph $G = (V, E)$ is called a matching. M is a matching of $U \subset V$ if every vertex in U is incident with an edge in M. The vertices in U are then called matched by M. Vertices not incident with any edge of M are unmatched. For a graph G let m_i denote the number of matchings in G consisting of i edges, with $m_0 \equiv 1$. Define the matching polynomial of G by

$$\mu_G(x) = \sum_{i\geq0} x^{n-2i} (-1)^i m_i \tag{5.18}$$

where n is the number of vertices of the graph.

The matching polynomial has an important application in statistical mechanics [61]. More precise, it was used in order to investigate phase transitions in a so called monomer-dimer configuration. This configuration consists of a number of nonoverlapping dimers, that is a matching, and monomers, that are unmatched vertices, in a graph. The main theorem of [61] is the following.

Theorem 5.12 ([61]). For any graph G, $\mu_G(x)$ has only real roots.

Proof. The idea is to establish first a recurrence relation satisfied by μ_G. For any $v \in V$ we have

$$\mu_G(x) = x\mu_{G\setminus\{v\}}(x) - \sum_{(u,v)\in E} \mu_{G\setminus\{u,v\}}(x) \tag{5.19}$$

where $G\setminus\{v\}$ denotes the vertex-deleted subgraph of G. First assume that G is a complete graph on n vertices. By the induction hypothesis, every graph H with at most $n-1$ vertices fulfills

(1) $\mu_H(x)$ is real-rooted with all roots distinct

(2) For every $v \in H$, $\mu_{H\setminus\{v\}}(x)$ interlaces $\mu_H(x)$

We will show in the following that the properties (1) and (2) must also be satisfied by G. Fix a vertex $v \in G$ and denote by $\lambda_{n-1} < ... < \lambda_1$ the roots of $\mu_{G\setminus\{v\}}$. By the induction hypothesis, we know that $\mu_{G\setminus\{u,v\}}$ strictly interlaces $\mu_{G\setminus\{v\}}$. Since each of these polynomials is monic, this implies that $\mu_{G\setminus\{u,v\}}(\lambda_1) > 0$. Further, as each interval $(\lambda_i, \lambda_{i+1})$ contains exactly one root of each $\mu_{G\setminus\{u,v\}}$, it follows

$$\text{sign}[\mu_{G\setminus\{u,v\}}(\lambda_i)] = (-1)^{i+1} \tag{5.20}$$

for $i = 1, ..., n-1$ and all $(u, v) \in E$. This implies that the polynomial

$$r(x) = \sum_{(u,v)\in E} \mu_{G\setminus\{u,v\}}(x) \tag{5.21}$$

must also alternate sign at λ_i. Using the recurrence relation (5.19), we have

$$\mu_G(\lambda_i) = \lambda_i\mu_{G\setminus\{v\}}(\lambda_i) - r(\lambda_i) = -r(\lambda_i) \tag{5.22}$$

and thus we obtain $\text{sign}[\mu_G(\lambda_i)] = (-1)^i$. The intermediate value theorem implies that μ_G has at least one root in each interval $(\lambda_i, \lambda_{i+1})$, what yields that μ_G has at least $n-2$ distinct roots. Since $\mu_G(x) < 0$ and $\mu_G(x) \to \infty$ for $x \to \infty$, we must also have $\mu_G)(\lambda_0) = 0$ for some $\lambda_0 > \lambda_1$. The same argument provides the existence of another root $\lambda_n < \lambda_{n-1}$, hence in total there are n distinct real roots, which are strictly interlaced by the roots of $\mu_{G\setminus\{v\}}$. □

The proof of the following theorem is in many parts equal to the proof of Theorem 5.12, that is, relies heavily on the recurrence relation (5.19). Therefore we will not present it explicitly here.

Theorem 5.13 ([61]). Let G be a graph with maximal degree d. Then all (real) roots of the matching polynomial $\mu_G(x)$ have absolute value at most $2\sqrt{d-1}$.

Definition 5.14. Let $G = (V, E)$ be a graph. A walk in G is a sequence of vertices $(v_1, ..., v_n)$ such that each consecutive pair (v_{i-1}, v_{i+1}) is an edge in G. A walk is called simple, if all vertices are distinct and non-backtracking if $v_{i-1} \neq v_{i+1}$ for all i. Further we say that a walk $\tilde{\omega}$ is a continuation of another walk ω, if it is obtained by adding a single vertex to ω, i.e., $\omega = (v_1, ..., v_n)$ and $\tilde{\omega} = (v_1, ..., v_n, v_{n+1})$ for some $v_{n+1} \in V(G)$. The universal covering graph T of G is a graph constructed as follows. First, choose an arbitrary vertex $v \in V(G)$ as a starting point. Each vertex of T is a non-backtracking walk that begins from v. Two vertices are adjacent, if one of the corresponding walks is a continuation of the other.

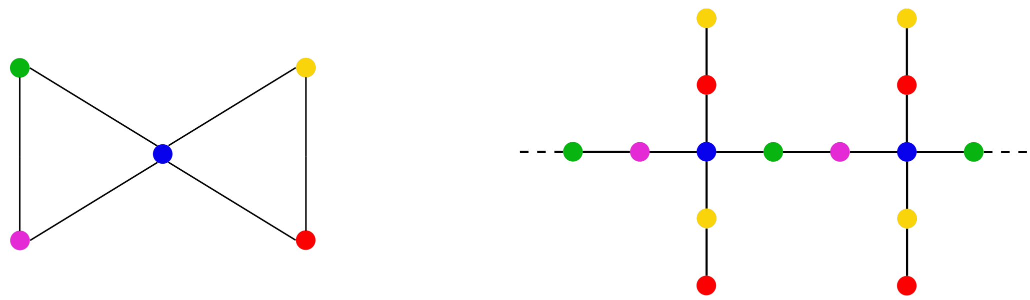


Figure 5.3: A graph and its universal covering graph. As starting point the blue vertex was chosen. For each walk in the graph at the left hand side which ends at a vertex of color c, we add a vertex at the universal covering graph in the appropriate color. Clearly, the walk (b, r, y) is a continuation of the walk (b, r), hence the corresponding vertices in the universal covering graph are connected by an edge.

A similar concept to the universal covering graph is that of the so called path tree, first introduced in [62].

Definition 5.15. Given a graph G and a vertex $v \in V(G)$, the path tree $P(G, v)$ contains one vertex for every simple walk in G beginning at u. Two vertices in $P(G, v)$ are adjacent, ig the simple walk corresponding to one is a continuation of the simple walk corresponding to the other.

From the definition it is clear that the path tree as well as the universal cover are graphs with an infinite number of vertices. Hence, the corresponding adjacency matrix is infinite dimensional symmetric matrix. The spectral radius of a matrix of this type can be defined as

$$\rho(T) := \sup_{||x||_2=1} ||A_T\, x||_2 \qquad (5.23)$$

where $|| \cdot ||_2$ denotes, in the case of existence, the 2-norm and A_T the adjacency matrix of T. The following theorem relates the roots of the matching polynomial of a graph with the spectral radius of its path tree.

Theorem 5.16 ([63]). Let $G = (V, E)$ be a graph and denote by $P(G, u)$ a path tree of G for $u \in V$. Then the matching polynomial of G divides the characteristic polynomial of the adjacency matrix of $P(G, u)$. In particular, all of the roots of $\mu_G(x)$ are real and have absolute value at most $\rho(P(G, u))$.

Theorem 5.17. Let $G = (V, E)$ be a graph and T its universal cover. Then the roots of $\mu_G(x)$ are bounded in absolute value by $\rho(T)$.

Proof. Let $v \in V$ and let P be the path tree rooted at the vertex v. Since the simple walks that correspond to the vertices of the path three P are nonbacktracking walks, we have that P is a finite induced subgraph of the universal cover T. Hence A_P, the adjacency matrix of P, is a finite submatrix of A_T. By virtue of Theorem 5.16, the roots of μ_G are bounded by

$$||A_P||_2 = \sup_{||x||_2=1} ||A_P\, x|| \leq \sup_{\substack{||y||_2=1 \\ \text{supp}(y)\subset P}} ||A_T\, y||_2 = \sup_{||y||_2=1} ||A_T\, y||_2 = \rho(T) \tag{5.24}$$

Thus we have that the roots of μ_G are bounded by $\rho(T)$. □

The following theorem is due to Godsil and Gutman and relates the expected characteristic polynomial over uniformly random signings of the adjacency matrix of a graph to its matching polynomial.

Theorem 5.18 ([62, 28]). Let $G = (V, E)$ be a graph, $m = |E|$, $s \in \{\pm 1\}^m$ a signing of G and $f_s(x) := \det[x\,\mathbb{1} - A_s]$. Then

$$\mathbb{E}_{s\in\{\pm1\}^m}\,[f_s(x)] = \mu_G(x) \tag{5.25}$$

Proof. If $S \subset \{1, ..., n\}$ we write $\mathfrak{S}(S)$ for the group of permutations on the set S. Further we write $|\pi|$ for the number of inversions of $\pi \in \mathfrak{S}(S)$, i.e., the number of pairs (i, j) with $i < j$, such that $\pi(i) < \pi(j)$. In particular, $(-1)^{|\pi|}$ is the signature of the permutation π. If $n = |V|$, we have $A_s \in \mathrm{M}_n(\mathbb{R})$ and an expansion of the determinant as a sum over permutations $\sigma \in \mathfrak{S}_n$ yields

$$\begin{aligned}
\mathbb{E}_s[\det(x\mathbb{1} - A_s)] &= \mathbb{E}_s\left[\sum_{\sigma\in\mathfrak{S}_n} (-1)^{|\sigma|} \prod_{i=1}^{n} (x\,\mathbb{1} - A_s)_{i,\sigma(i)}\right] \\
&= \sum_{k=0}^{n} x^{n-k}(-1)^k \sum_{\substack{S\subset[n] \\ |S|=k}} \sum_{\pi\in\mathfrak{S}(S)} \mathbb{E}_s\left[(-1)^{|\pi|} \prod_{i\in S} (A_s)_{i,\pi(i)}\right] \\
&= \sum_{k=0}^{n} x^{n-k}(-1)^k \sum_{\substack{S\subset[n] \\ |S|=k}} \sum_{\pi\in\mathfrak{S}(S)} \mathbb{E}_s\left[(-1)^{|\pi|} \prod_{i\in S} s_{i,\pi(i)}\right]
\end{aligned} \tag{5.26}$$

Since the s_{ij} are chosen independently, i.e., $\mathbb{P}(s_{ij} = 1) = \mathbb{P}(s_{ij} = -1)$, we have $\mathbb{E}(s_{ij}) = 0$. Hence, it is sufficient to only consider products that contain even powers of s_{ij}. On the level of permutations this means, that we can restrict to those π, whose orbit is of size two. But there are exactly the perfect matchings on S. Since there are no perfect matchings if $|S|$ is odd, we only consider the case where $|S|$ even. Here, each matching consists of $|S|/2$ disjoint transpositions. Since $\mathbb{E}_s(s_{ij}^2) = 1$, this yields

$$\mathbb{E}_s[\det(x\,\mathbb{1} - A_s)] = \sum_{\substack{k=0 \\ k \text{ even}}} x^{n-k} \sum_{|S|=k} \sum_{\substack{\text{matchings} \\ \pi \text{ on} S}} (-1)^{|S|/2} = \mu_G(x) \tag{5.27}$$

□

As we will see, one important step towards a proof of Theorem 5.10 is Theorem 5.22, which heavily relies on the concepts of interlacing families as well as real stability. Consequently, we will benefit in the following from the observations already made in Section 4. In particular, we directly obtain from Lemma 4.15 the following

Corollary 5.19. For all $a, b \geq 0$ and variables x, y, the operator $T := 1 + a\partial_x + b\partial_y$ preserves real stability.

Similar to the proof of Theorem 4.30, we can use the operators introduced in Corollary in order to generate real stable polynomials, given a real stable polynomial. In fact, we will see, that we can generate the expected characteristic polynomial by operations of this kind. Further we have seen in Lemma 4.15, that also the operation of restriction preserves real stability. More formal this means, that the operator $Z_{z_k} : \mathbb{R}[z_1, ..., z_n] \to \mathbb{R}[z_1, ..., z_{k-1}, z_{k+1}, ..., z_n]$ with $p \mapsto p|_{z_k=0}$ preserves real stability.

Lemma 5.20. Let $A \in \mathrm{GL}(n, \mathbb{C})$, $p \in [0, 1]$ and $u, v \in \mathbb{C}^n$. Then, the following identity holds

$$\begin{aligned} Z_x Z_y (1 + p\partial_x + (1-p)\partial_y) \det(A + xuu^\top + yvv^\top) \\ = p \det(A + uu^\top) + (1-p) \det(A + vv^\top) \end{aligned} \tag{5.28}$$

Proof. If we apply Theorem 4.4 to the function $\det(A + tuu^\top)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \det(A + tuu^\top) = \det(A)\,(u^\top A^{-1} u) \tag{5.29}$$

First, consider the left-hand side of (5.28) without $Z_x Z_y$. Using that $1 = p + (1-p)$ we obtain

$$\begin{aligned} (1 + p\partial_x + (1-p)\partial_y) \det(A + xuu^\top + yvv^\top) = \det(A + xuu^\top + yvv^\top) \\ + p \det(A + yvv^\top)(u^\top A^{-1} u) + (1-p) \det(A + xuu^\top)(v^\top A^{-1} v) \end{aligned} \tag{5.30}$$

An application of the operator $Z_x Z_y$ on (5.30) yields

$$\begin{aligned} Z_x Z_y (1 + p\partial_x + (1-p)\partial_y) \det(A + xuu^\top + yvv^\top) \\ = \det(A)\,[1 + pu^\top A^{-1} u + (1-p) v^\top A^{-1} v] \\ = p \det(A)(1 + u^\top A^{-1} u) + (1-p) \det(A)(1 + v^\top A^{-1} v) \\ = p \det(A + uu^\top) + (1-p) \det(A + vv^\top) \end{aligned} \tag{5.31}$$

where we have used the matrix determinant lemma in the last step. This proves the claim. □

Lemma 5.21. Let $u_1, , ..., u_m, v_1, ..., v_m \in \mathbb{R}^n$ and $p_1, ..., p_m \in [0, 1]$. Further let $A \in \mathrm{M}_n(\mathbb{C})$ be a positive semidefinite hermitian matrix. Then the univariate polynomial

$$P(x) := \sum_{S \subset [m]} \prod_{i \in S} p_i \prod_{i \notin S} (1 - p_i) \det \left[x\mathbb{1} + A + \sum_{i \in S} u_i u_i^\top + \sum_{i \notin S} v_i v_i^\top \right] \tag{5.32}$$

is real rooted.

Proof. In order to prove that the univariate polynomial $P(x)$ is real rooted, we consider the multivariate polynomial

$$Q(x, y_1, ..., y_m, z_1, ..., z_m) := \det \left[x\mathbb{1} + A + \sum_{i \in S} y_i u_i u_i^\top \sum_{i \notin S} z_i v_i v_i^\top \right] \tag{5.33}$$

Since operators of the form $vv^\top$ for $v \in \mathbb{C}^n$ are positive semidefinite, we can conclude from Lemma 4.14 and the fact, that specifying variables to real numbers preserves real stability (cf. Lemma 4.15) that Q is real stable. The idea is now the following. If we can show that P admits a representation as

$$P(x) = \left[\prod_{i=1}^{m} Z_{y_i} Z_{z_i} T_i\right] Q(x, y_1, ..., y_m, z_1, ..., z_m) \tag{5.34}$$

where $T_i = 1 + p_i\partial_{y_i} + (1 - p_i)\partial_{z_i}$, by virtue of Lemma 4.15 and Corollary 5.19 we have that Q is real stable. We will prove (5.34) by induction. In particular, we show that for any $k \in \mathbb{N}$ we have

$$\begin{aligned} &\left[\prod_{i=1}^{k} Z_{y_i} Z_{z_i} T_i\right] Q(x, y_1, ..., y_m, z_1, ..., z_m) = \\ &\sum_{S\subset[k]} \prod_{i\in S} p_i \prod_{i\notin S}(1 - p_i) \det\left[x\mathbb{1} + A + \sum_{i\in S} u_i u_i^\top + \sum_{i\notin S} v_i v_i^\top + \sum_{i>k}(y_i u_i u_i^\top + z_i v_i v_i^\top)\right] \end{aligned} \tag{5.35}$$

The case $k = 0$ is trivial, since the right hand side of (5.35) coincides with the definition of Q. For the induction step suppose that (5.35) holds for $k \in \mathbb{N}$. Since Q is a polynomial, it is a C^∞-function and by the theorem of Schwarz, the partial derivatives commute. Therefore we have

$$\begin{aligned} &\left[\prod_{i=1}^{k+1} Z_{y_i} Z_{z_i} T_i\right] Q(x, y_1, ..., y_m, z_1, ..., z_m) = Z_{y_{k+1}} Z_{z_{k+1}} T_{k+1} \left[\prod_{i=1}^{k} Z_{y_i} Z_{z_i} T_i\right] Q = \\ &Z_{y_{k+1}} Z_{z_{k+1}} T_{k+1}\Big(\sum_{S\subset[k]} \prod_{i\in S} p_i \prod_{i\notin S}(1 - p_i) \det\left[B + y_{k+1} u_{k+1} u_{k+1}^\top + z_{k+1} v_{k+1} v_{k+1}^\top\right]\Big) \end{aligned} \tag{5.36}$$

where we have defined $B := x\mathbb{1} + A + \sum_{i\in S} u_i u_i^\top + \sum_{i\notin S} v_i v_i^\top + \sum_{i>k+1}(y_i u_i u_i^\top + z_i v_i v_i^\top)$. Now, we apply Lemma 5.20 to each term that appears in the sum over the subsets $S \subset [k]$ what yields in the case of $k = m$ exactly the claimed identity. This provides that $P(x)$ is real stable and since it is univariate, it is real-rooted. □

Theorem 5.22 ([28]). Let $G = (V, E)$ be a graph, A_s its signed adjacency matrix and $f_s(x) := \det[\mathbb{1}x - A_s]$ its characteristic polynomial. Then

$$P(x) = \sum_{s\in\{\pm1\}^m} \left[\prod_{i:s_i=1} p_i\right] \left[\prod_{i:s_i=-1} (1 - p_i)\right] f_s(x) \tag{5.37}$$

is a real-rooted polynomial for all values of $p_1, ..., p_m \in [0, 1]$.

Proof. Let $v \in V$ be a vertex and denote by d_v its degree. Further define $d := \max_{v\in V} d_v$. First observe that the claim that (5.37) is real-rooted is equivalent to the claim that the polynomial

$$\sum_{s\in\{\pm1\}^m} \prod_{i:s_i=1} \prod_{i:s_i=-1} \det(x\mathbb{1} + d\mathbb{1} - A_s) \tag{5.38}$$

is real-rooted, as their roots only differ by d. The matrix $d\mathbb{1} - A_s$ can be seen as the Laplacian of the graph G plus a non-negative diagonal matrix, i.e., the matrix $\delta = (d -$

$d_v)e_v e_v^\top$. For each edge $(u,v) \in E$, define the rank one matrices, which are labeled by the edge (u,v) as well as its signing

$$L^1_{u,v} = (e_u - e_v)(e_u - e_v)^\top \ , \ L^{-1}_{u,v} = (e_u + e_v)(e_u + e_v)^\top \tag{5.39}$$

Consider now a signing $s \in \{\pm 1\}^{|V|}$ of the graph G, and let $s(u,v)$ denote the sign which it asserts to the edge (u,v). By the definition of d and δ we obtain

$$d\,\mathbb{1} - A_s = \sum_{(u,v)\in E} L^{s(u,v)}_{u,v} + \delta \tag{5.40}$$

Since δ is a positive semidefinite matrix, we can conclude that also $d\,\mathbb{1} - A_s$ is positive semidefinite. Define $a_{u,v} := (e_u - e_v)$ and $b_{u,v} = e_u + e_v$ and rewrite (5.40) in terms of those

$$\sum_{s\in\{\pm1\}^m} \prod_{i:s_i=1} \prod_{i:s_i=-1} \det\left[x\,\mathbb{1} + A + \sum_{s(u,v)=1} a_{u,v}a_{u,v}^\top + \sum_{s(u,v)=-1} b_{u,v}b_{u,v}^\top\right] \tag{5.41}$$

That the polynomial (5.41) is real-rooted follows from Lemma 5.21, what implies the real-rootedness of (5.37). Hence the claim follows. □

Theorem 5.23 ([28]). Let $G = (V,E)$ be a graph, A_s its signed adjacency matrix for some signing $s \in \{\pm1\}^{|V|}$ and $f_s(x) := \det[\mathbb{1}\,x - A_s]$ its characteristic polynomial. The polynomials $\{f_s\}_{s\in\{\pm1\}^m}$ form an interlacing family.

Proof. First observe, that for every $0 \le k \le m-1$, every assignment $s_i \in \{\pm1\}$ with $1 \le i \le k$ and every $\lambda \in [0,1]$ the polynomial

$$\lambda f_{s_1,\dots,s_k,1}(x) + (1-\lambda) f_{s_1,\dots,s_k,-1}(x) \tag{5.42}$$

is real rooted. This follows from Theorem 5.22, if we set in (5.37) $p_{k+1} = \lambda$, $p_{k+1},\dots,p_m = \frac{1}{2}$ and $p_i = \frac{1+s_i}{2}$ for $1 \le i \le k$. Since this holds for every value of $\lambda \in [0,1]$ and every $0 \le k \le m-1$, we can conclude that also $\sum_{s\in\{\pm1\}^m} \lambda_s f_s$ is real-rooted, where $\lambda_s \in [0,1]$. By virtue of Lemma 4.23 we can conclude that the polynomials $\{f_s\}_{s\in\{\pm1\}^m}$ have a common interlacing. □

Theorem 5.24 ([28]). Let $G = (V,E)$ be a graph with adjacency matrix A and universal cover T. Then there exists a signing s of A such that all eigenvalues of A_s are bounded by $\rho(T)$. If G is d-regular, there exists a signing s such that all eigenvalues of A_s are at most $2\sqrt{d-1}$.

Proof. Denote by $m = |E|$. The eigenvalues of A_s are given by the roots of the polynomial $f_s(x) = \det(x\,\mathbb{1} - A_s)$. By Theorem 5.23 we know that the polynomials $\{f_s\}_{s\in\{\pm1\}^m}$ form an interlacing family. Further we have $\mathbb{E}_{s\in\{\pm1\}^m}(f_s(x)) = \mu_G(x)$ by Theorem 5.18 and we can bound the largest root of $\mu_G(x)$ by $\text{maxroot}(\mu_G) \le \rho(T)$. Then, Theorem 4.22 guarantees the existence of a signing $s \in \{\pm1\}^m$ such that $\text{maxroot}[f_s(x)] \le \rho(T)$. For the second claim observe that a d-regular graph is of maximal degree d. Hence we can apply Theorem 5.13 which yields that $\text{maxroot}(\mu_G) \le 2\sqrt{d-1}$. Using Theorem 4.22 and the same argumentation as above, we obtain the existence of a signing $s \in \{\pm1\}^m$ such that the eigenvalues of A_s are at most $2\sqrt{d-1}$. □

Before we are able to prove the main theorems, observe that if G is a complete d-regular graph, all its nontrivial eigenvalues are zero. This can be easily seen from the fact, that the corresponding adjacency matrix is of rank two and hence, there can be only two nonvanishing eigenvalues. But these both eigenvalues have to be $\pm d$, thus trivial.

Proof of Theorem 5.10. By the previous discussion, we know that the complete bipartite graph of degree d is Ramanujan. Further, the 2-lift of a bipartite graph is again bipartite, hence its eigenvalues are symmetric about 0. In particular, the 2-lift of a d-regular bipartite graph is a d-regular bipartite graph. It remains to show that there is a 2-lift of G, in which every nontrivial eigenvalue is at most $2\sqrt{d-1}$. But this is exactly the content of Theorem 5.24. □

5.2 Bourgain-Tzafriri conjecture

In this section we are going to show that the Kadison-Singer problem is equivalent to the Bourgain-Tzafriri conjecture, that is a strong form of the restricted invertibility problem, posed in [64]. There, Bourgain and Tzafriri proved the following result known as the restricted invertibility theorem

Theorem 5.25 ([64]). Let $T : \ell_2^n \to \ell_2^n$ be a linear operator with $||Te_i|| = 1$ for $1 \leq i \leq n$. Then there exist universal constants $A, c > 0$ and a subset $\sigma \subset \{1, ..., n\}$ with $|\sigma| \geq cn/||T||^2$ such that for all $j = 1, ..., n$ and all choices of $\{a_j\}_{j\in\sigma} \subset \mathbb{R}$ we have

$$||\sum_{j\in\sigma} a_j Te_j||^2 \geq A \sum_{j\in\sigma} |a_j|^2 \tag{5.43}$$

Theorem 5.25 gave rise to the following conjecture.

Conjecture 5.26 (BT). There is a universal constant $A > 0$ so that there exists for any $B > 1$ a natural number $r = r(B)$ such that the following holds. For any $n \in \mathbb{N}$ and $T : \ell_2^n \to \ell_2^n$ linear with $||T|| \leq B$ and $||Te_i|| = 1$ there exists a partition $\{A_j\}_{j=1}^r$ of $\{1, ..., n\}$ such that for all $j = 1, ..., r$ and all choices of $\{a_j\}_{i\in A_j}$ we have

$$||\sum_{i\in A_j} a_i Te_i||^2 \geq A \sum_{i\in A_j} |a_i|^2 \tag{5.44}$$

Frequently Conjecture 5.26 is also called strong Bourgain-Tzafriri conjecture, since there exists also a weakening of it. In the weak Bourgain-Tzafriri conjecture one also allows A to depend upon the norm of the operator T. In order to show that the strong Bourgain-Tzafriri conjecture as well as the weak Bourgain-Tzafriri conjecture are equivalent to the Kadison-Singer conjecture we first need a further conjecture, the so called Casazza-Tremain conjecture, equivalent to both above.

Conjecture 5.27 (CT). For $n \in \mathbb{N}$ let $T : \ell_2^n \to \ell_2^n$ be linear with $||Te_i|| = 1$ for all $i = 1, ..., n$ and $||T|| \leq 2$. Then there exists a universal constant A and $r \in \mathbb{N}$ such that there is a partition $\{A_j\}_{j=1}^r$ of $\{1, ..., n\}$ so that for all $j = 1, ..., r$ and all $\{a_i\}_{i\in A_j} \subset \mathbb{R}$ we have

$$||\sum_{i\in A_j} a_i Te_i||^2 \geq A \sum_{i\in A_j} |a_i|^2 \tag{5.45}$$

Theorem 5.28. The following are equivalent:

(a) The Kadison-Singer Problem

(b) The strong Bourgain-Tzafriri Conjecture

(c) The weak Bourgain-Tzafriri Conjecture

(d) The Casazza-Tremain Conjecture

Proof. We will prove the following chain of implications (a)⇒ (b) ⇒ (c) ⇒ (d) ⇒ (a). We first show (d)⇒ (a). By Theorem 3.33 it is sufficient to show, that the Casazza-Tremain Conjecture implies the Weaver Conjecture. For this purpose, let r, A satisfy (CT). Fix $0 < \delta \leq 3/4$ and let P be an orthogonal projection on ℓ_2^n such that $\delta(P) \leq \delta$. Clearly one has

$$||Pe_i||^2 = \langle Pe_i, e_i \rangle = P_{ii} \leq \max_{i,j} P_{ij} = \delta \tag{5.46}$$

This implies $||(\mathbb{1} - P)e_i||^2 \geq 1 - \delta \geq 1/4$. Define the operator $T : \ell_2^n \to \ell_2^n$ by its action on the standard basis e_i and then extend by linearity to the whole space i.e., $Te_i = (\mathbb{1} - P)e_i / ||(\mathbb{1} - P)e_i||$. For any $\{a_i\}_{i=1}^n$ we have

$$||\sum_{i=1}^n a_i Te_i||^2 = ||\sum_{i=1}^n \frac{a_i}{||(\mathbb{1} - P)e_i||}(\mathbb{1} - P)e_i||^2 \tag{5.47}$$

$$\leq \sum_{i=1}^n |\frac{a_i}{||(\mathbb{1} - P)e_i||}|^2 \leq \sum_{i=1}^n |\frac{a_i}{1-\delta}|^2 \leq 4\sum_{i=1}^n |a_i|^2 \tag{5.48}$$

Hence $||Te_i|| = 1$ and $||T|| \leq 2$. By the CT conjecture, there exists a partition $\{A_j\}_{j=1}^r$ of $\{1, ..., n\}$ such that for $j = 1, ..., r$ and $\{a_i\}_{i=1}^n$ we have

$$||\sum_{i \in A_j} a_i Te_i||^2 \geq A \sum_{i \in A_j} |a_i|^2 \tag{5.49}$$

Thus we have

$$||\sum_{i \in A_j} a_i (\mathbb{1} - P)e_i||^2 = \sum_{i \in A_j} a_i \cdot ||(\mathbb{1} - P)e_i|| \cdot (Te_i)||^2 \tag{5.50}$$

$$\geq A \sum_{i \in A_j} |a_i|^2 \cdot ||(\mathbb{1} - P)e_i||^2 \geq A \sum_{i \in A_j} |a_i|^2 \delta \geq \frac{A}{4} \sum_{i \in A_j} |a_i|^2 \tag{5.51}$$

Clearly we have $P(\mathbb{1} - P) = 0$, and consequently $||Pe_i + (\mathbb{1} - P)e_i||^2 = ||Pe_i||^2 + ||(\mathbb{1} - P)e_i||^2$. It follows that for all $\{a_i\}_{i \in A_j}$ we have

$$\sum_{i \in A_j} |a_i|^2 = \sum_{i \in A_j} ||a_i e_i||^2 = \sum_{i \in A_j} ||a_i Pe_i + a_i(\mathbb{1} - p)e_i|| \tag{5.52}$$

$$||\sum_{i \in A_j} a_i Pe_i||^2 + ||\sum_{i \in A_j} a_i (\mathbb{1} - P)e_i||^2 \geq ||\sum_{i \in A_j} a_i Pe_i||^2 + \frac{A}{4} \sum_{i \in A_j} |a_i|^2 \tag{5.53}$$

where the inequality follows by (5.51). For $\mathbb{R}^n \ni f = \sum_{i=1}^n a_i e_i$ and by virtue of the estimation (5.53) we obtain

$$||PQ_{A_j}f||^2 = ||\sum_{i\in A_j} a_i Pe_i||^2 \leq \left(1-\frac{A}{4}\right)\sum_{i\in A_j}|a_i|^2 \tag{5.54}$$

Since P is an orthogonal projection and the C^*-property of the norm, we end up with

$$||Q_{A_j}PQ_{A_j}|| = ||\left(PQ_{A_j}\right)^* PQ_{A_j}|| = ||PQ_{A_j}||^2 \leq 1-\frac{A}{4} \tag{5.55}$$

Hence Weaver's conjecture 3.32 holds and by virtue of Theorem 3.33 we can conclude that (d) $\Rightarrow$ (a). The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are clear, since we also allow that the constant A depends on the norm of the operator T. □

One can also formulate the Paving Conjectures in terms of Toeplitz operators and by this means connect it to a fundamental problem in Harmonic analysis.

Definition 5.29. Let $\phi \in L^\infty([0,1])$. The Toeplitz operator corresponding to ϕ is defined as

$$T_\phi : L^2([0,1]) \to L^2([0,1]) \quad , \quad f \mapsto f\cdot\phi \tag{5.56}$$

Further, if $I\subset\mathbb{Z}$, we denote by $S(I)$ the $L^2([0,1])$-closure of the span exponential functions with frequencies taken from I, i.e.,

$$S(I) = \overline{\text{span}\{e^{2\pi int}\}_{n\in I}} \tag{5.57}$$

Lemma 5.30. Let $E\subset[0,1]$ be a measurable subset and $A\subset\mathbb{Z}$. For every $f\in L^2([0,1])$ we have

$$||P_EQ_Af||^2 = \mu(E)\,||Q_Af||^2 + \langle Q_A(P_E-\mathcal{D}(P_E))Q_Af, f\rangle \tag{5.58}$$

where Q_A denotes the orthogonal projection of $L^2([0,1])$ onto $S(A)$

Proof. For $L^2([0,1]) \ni f = \sum_{n\in\mathbb{Z}} a_n e^{2\pi int}$ it follows that

$$||P_EQ_Af||^2 = \langle P_EQ_Af, P_EQ_Af\rangle = \langle\sum_{n\in A} a_nP_E\left(e^{2\pi int}\right), \sum_{m\in A} P_E\left(e^{2\pi imt}\right)\rangle \tag{5.59}$$

Ordering the terms with respect to their indices and inserting $P_E(f) = \chi_E\cdot f$ and using $P_E^2 = P_E$ yields

$$||P_EQ_Af||^2 = \sum_{n\in A}|a_n|^2\cdot||\chi_E\cdot e^{2\pi int}||^2 + \sum_{n\neq m} a_n\overline{a}_m\langle P_Ee^{2\pi int}, e^{2\pi imt}\rangle \tag{5.60}$$

$$= \mu(E)\sum_{n\in A}|a_n|^2 + \langle(P_E-\mathcal{D}(P_E))\sum_{n\in A}a_ne^{2\pi int}, \sum_{n\in A}a_ne^{2\pi int}\rangle \tag{5.61}$$

$$= \mu(E)\,||Q_Af||^2 + \langle Q_A(P_E-\mathcal{D}(P_E))Q_Af, f\rangle \tag{5.62}$$

□

Conjecture 5.31 (Toep). Let $E \subset [0,1]$ be a measurable set and $\epsilon > 0$ given. Then there exists a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ such that one has for $f \in S(A_j)$ and all $j = 1, ..., r$

$$(1-\epsilon)\,\mu(E)\,||f||^2 \le ||P_E(f)||^2 \le (1+\epsilon)\,\mu(E)\,||f||^2 \tag{5.63}$$

Theorem 5.32. The following are equivalent:

(a) The Toeplitz Conjecture

(b) All Toeplitz operators satisfy the Paving Conjecture

(c) For every measurable set $E \subset [0,1]$, the Toeplitz operator P_E satisfies the Paving Conjecture

Proof. We will prove the following equivalences (a) ⇔ (b) ⇔ (c). We start with (a) ⇔ (c). From Lemma 5.30 we can conclude that the Toeplitz conjecture hold if and only if for $\epsilon > 0$ arbitrary, we can find a partition $\{A_j\}_{j=1}^r$ such that

$$(1-\epsilon)\,\mu(E)\,||Q_{A_j}f||^2 \le \mu(E)\,||Q_{A_j}f||^2 + \langle Q_{A_j}\left(P_E - \mathcal{D}\left(P_E\right)\right)Q_{A_j}f, f\rangle \tag{5.64}$$

$$\le (1+\epsilon)\,\mu(E)\,||Q_{A_j}f||^2 \tag{5.65}$$

for all $j \in \{1, ..., r\}$ and all $f \in L^2(0,1)$. Further, by adding $\pm\mu(E)\,||Q_{A_j}f||^2$ to both sides of (5.64) , one finds that equation (5.64) is equivalent to

$$|\langle Q_{A_j}\left(P_E - \mathcal{D}\left(P_E\right)\right)Q_{A_j}f, f\rangle| \le \epsilon\mu(E)\,||Q_{A_j}f||^2 \tag{5.66}$$

By the self-adjointness of the operator $Q_{A_j}\left(P_E - \mathcal{D}\left(P_E\right)\right)Q_{A_j}$, we can rewrite (5.66) equivalently as $||Q_{A_j}\left(P_E - \mathcal{D}\left(P_E\right)\right)Q_{A_j}|| \le \epsilon\mu(E)$. But this means that the operator P_E is pavable. It remains to prove (b) ⇔ (c). First notice [65], that the class of pavable operators, i.e., the class of operators satisfying the paving conjecture, is a closed subspace of $B(\ell_2(\mathbb{N}))$. Further, the class of Toeplitz operators is contained in the closure of the linear span of the Toeplitz operators that are of the form P_E. This means, that we can uniformly approximate an arbitrary bounded measurable function on $[0,1]$ by simple functions. Hence we have (b)⇔ (c). □

Chapter 6

Conclusion

In this thesis we gave an introduction into the mathematical framework necessary to formulate the Kadison-Singer problem, its equivalent formulations and its solution. In particular, we summarized the main ingredients from the fields of topology, operator algebras as well as frame theory. Afterwards we used the introduced concepts to formulate the original form of the Kadison-Singer problem and discussed how the question evolves in dependence of new insights. In particular, we formulated the finite dimensional case in a way appropriate for a C^*-algebraic treatise and showed the uniqueness of extensions in this case. In addition, we also presented the approach of Kadison and Singer in order to construct a counterexample for the case of the continuous maximal abelian subalgebra, identified with $L^\infty(0,1)$. This includes a detailed analysis of von Neumann's diagonal processes. We then proceeded by introducing the paving conjecture in its infinite as well as in its finite formulation and proving the equivalence of both to the Kadison-Singer problem. We finished this chapter with a presentation of Weaver's conjecture. In Chapter 4 we presented the proof of Weaver's conjecture due to the probabilistic result of Marcus, Spielman and Srivastava. This comprises the concepts of interlacing families, the mixed characteristic polynomial, real stable polynomials as well as the multivariate barrier argument. We concluded with Chapter 5 by presenting two problems closely related to the solution of the Kadison-Singer problem. In particular, we discussed how the introduced concepts can be utilized to solve a question about the existence of Ramanujan graphs.

As a question for further research, it turns out that the problem of Ramanujan graphs, i.e., the question of existence, is similar to an open problem in quantum information science, namely to the existence of symmetric, informationally complete, positive operator-valued measures, or in short form SIC-POVM. Here a SIC-POVM is a collection of self-adjoint rank-1 projectors $(E_k)_{k=1}^{d^2} \subset \mathrm{M}_d(\mathbb{C})$ with $\mathrm{span}\{(E_k)\} = \mathrm{M}_d(\mathbb{C})$ and $\mathrm{tr}[E_iE_j] = (d+1)^{-1}(d\delta_{ij}+1)$. For small dimensions, it is easy to construct SIC-POVMs. For instance for the case $d=2$ we can take the vectors

$$|0\rangle \ , \ \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle \ , \ \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{\frac{2\pi i}{3}}|1\rangle \ , \ \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{\frac{4\pi i}{3}}|1\rangle \tag{6.1}$$

However, there is no construction procedure known manufacturing SIC-POVMs for any $d \in \mathbb{N}$, i.e., yielding an infinite family of SIC-POVMs. Exact expressions for SIC sets have been found for Hilbert spaces of all dimensions from $d = 2, ..., 53$ and in some higher dimensions as large as $d = 5799$. Therefore, it would be interesting whether it is possible to construct a SIC-POVM from a given Ramanujan graph, which would then provide a positive answer.

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Bachelorarbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt, sowie Zitate und Ergebnisse Anderer kenntlich gemacht habe.

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