Einstein-Podolsky-Rosen correlations and Bell correlations in the simplest scenario

Huangjun Zhu
(Joint work with Quan Quan, Heng Fan, and Wen-Li Yang)

Institute for Theoretical Physics, University of Cologne

PRA 95, 062111 (2017)

Introduction

- Steering is a nonclassical phenomenon that formalizes what Einstein called "spooky action at a distance". For a long time, it was studied under the name of Einstein-Podolsky-Rosen (EPR) paradox.
- It is a form of nonlocality that sits between entanglement and Bell nonlocality and that is intrinsically asymmetric.
- It can be characterized by a simple quantum information processing task, namely, entanglement verification with an untrusted party.
- It is useful in a number of applications, such as subchannel discrimination and one-sided device-independent quantum key distribution.

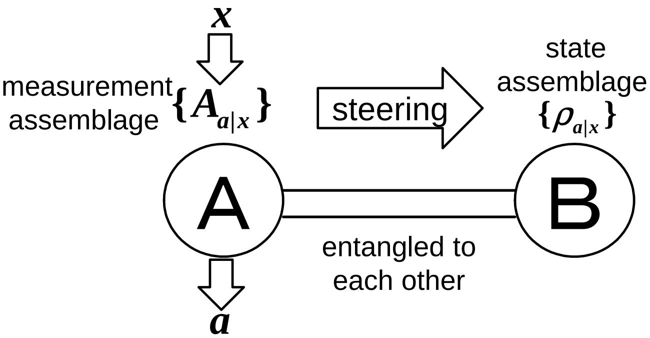

Figure: Steering scenario. Alice can affect Bob's state via her choice of the measurement according to the relation $\rho_{a|x} = \mathrm{tr}_A[(A_{a|x} \otimes 1)\rho]$. Entanglement is necessary but not sufficient for steering.

LHV model *vs.* LHV-LHS model

- Local hidden variable (LHV) model

$$p(a,b|x,y) = \sum_{\lambda} p_\lambda p(a|x,\lambda)p(b|y,\lambda).$$

$p(a|x,\lambda)$, $p(b|y,\lambda)$: arbitrary probability distributions,

- Local hidden variable-local hidden state (LHV-LHS) model[1],

$$p(a,b|x,y) = \sum_{\lambda} p_\lambda p(a|x,\lambda)p(b|y,\rho_\lambda).$$

$p(a|x,\lambda)$: arbitrary probability distributions, $p(b|y,\rho_\lambda) = \mathrm{tr}(\rho_\lambda B_b)$: probability distributions from Born rule.

The set of probability distributions $p(a,b|x,y)$ is EPR nonlocal (Bell) if it does not admit any LHV-LHS (LHV) model.

[1] Werner, PRA, 40, 4277 (1989); Wiseman et al., PRL **98**, 140402 (2007).

Simplest Bell scenario and Simplest Steering scenario

Simplest Bell scenario: Two dichotomic measurements for Alice and Bob, respectively.

The set of correlations is Bell nonlocal iff it violates the CHSH inequality.

Simplest steering scenario:

1. Two dichotomic measurements for Alice and Bob, respectively.
2. Two dichotomic measurements for Alice and one trine measurement for Bob.

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Which two-qubit states can generate EPR-nonlocal correlations in the simplest scenario?

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Measurement and state assemblages

- A positive-operator-valued measure (POVMs) is composed of a set of positive operators that sum up to the identity.
- A measurement assemblage $\{A_{a|x}\}$ is a collection of POVMs.
- Ensembles and state assemblages:

$$\rho_{a|x} = \mathrm{tr}[(A_{a|x} \otimes 1)\rho], \quad \sum_a \rho_{a|x} = \rho_B = \mathrm{tr}_A(\rho). \tag{1}$$

The set of unnormalized states $\rho_{a|x}$ for a given measurement x is an ensemble for ρ_B, and the whole collection $\{\rho_{a|x}\}$ a state assemblage

Steering and local hidden state model

- The assemblage $\{\rho_{a|x}\}$ admits a local hidden state model if

$$\rho_{a|x} = \sum_{\lambda} p(a|x,\lambda)\sigma_{\lambda} \quad \forall a, x, \tag{2}$$

  where $\{\sigma_{\lambda}\}$ is an ensemble for ρ_B and $p(a|x,\lambda)$ are a collection of stochastic maps.
- The assemblage $\{\rho_{a|x}\}$ is steerable it does not admit a local hidden state model.
- The state ρ is steerable from Alice to Bob if there exists a measurement assemblage for Alice such that the resulting state assemblage for Bob is steerable.

Restricted LHS model

- Let $\mathcal{V} \leq \mathcal{B}(\mathcal{H})$ be a subspace of the operator space. The assemblage $\{\rho_{a|x}\}_{a,x}$ admits a $\mathcal{V}$-restricted LHS model if

$$\mathrm{tr}(\Pi\rho_{a|x}) = \sum_{\lambda} p_\lambda p(a|x,\lambda)\,\mathrm{tr}(\Pi\rho_\lambda) \quad \forall\, \Pi \in \mathcal{V}.$$

  Otherwise, it is $\mathcal{V}$-steerable.

- Let $\mathcal{R}$ be the space spanned by all the effects B_b.

$$\{p(a,b|x,y)\} \text{ is EPR nonlocal} \Longleftrightarrow \{\rho_{a|x}\}_{a,x} \text{ is } \mathcal{R}\text{-steerable.}$$

- Consider the two-qubit state

$$\rho = \frac{1}{4}\left(I \otimes I + \boldsymbol{\alpha} \cdot \boldsymbol{\sigma} \otimes I + I \otimes \boldsymbol{\beta} \cdot \boldsymbol{\sigma} + \sum_{i,j=1}^{3} t_{ij}\sigma_i \otimes \sigma_j\right),$$

Alice and Bob can choose two projective measurements as described by $\{A_1, A_2\} = \{\boldsymbol{a}_1 \cdot \boldsymbol{\sigma}, \boldsymbol{a}_2 \cdot \boldsymbol{\sigma}\}$ and $\{B_1, B_2\} = \{\boldsymbol{b}_1 \cdot \boldsymbol{\sigma}, \boldsymbol{b}_2 \cdot \boldsymbol{\sigma}\}$.

- Assemble of Bob induced by Alice,

$$\rho_{\pm|m} = \frac{1}{4}\left[(1 \pm \alpha \cdot \boldsymbol{a}_m)I + \beta \cdot \boldsymbol{\sigma} \pm \gamma_m \cdot \boldsymbol{\sigma}\right],$$

where $\gamma_{mj} = \sum_{i=1}^{3} a_{mi} t_{ij}$.

- Assemblage after projection:

$$\tilde{\rho}_{\pm|m} = \frac{1}{4}\left[(1 \pm \alpha \cdot \boldsymbol{a}_m)I + \tilde{\beta} \cdot \boldsymbol{\sigma} \pm \tilde{\gamma}_m \cdot \boldsymbol{\sigma}\right],$$

where $\tilde{\beta}$ and $\tilde{\gamma}_m$ are the projection of and β and γ_m on the plane spanned by $\boldsymbol{b}_1, \boldsymbol{b}_2$.

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where $\tilde{\beta}$ and $\tilde{\gamma}_m$ are the projection of and β and γ_m on the plane spanned by $\boldsymbol{b}_1, \boldsymbol{b}_2$.

Necessary and sufficient steering criterion

- The state ρ is steerable under the measurement setting $\{\boldsymbol{a}_1 \cdot \boldsymbol{\sigma}, \boldsymbol{a}_2 \cdot \boldsymbol{\sigma}\}$ and $\{\boldsymbol{b}_1 \cdot \boldsymbol{\sigma}, \boldsymbol{b}_2 \cdot \boldsymbol{\sigma}\}$ iff the assemblage $\{\tilde{\rho}_{\pm|m}\}$ is steerable.
- Equivalently, the two effects $O_{+|1}$ and $O_{+|2}$ are not coexistent, where

$$O_{\pm|m} = \tilde{\rho}_{\mathrm{B}}^{-1/2} \tilde{\rho}_{\pm|m} \tilde{\rho}_{\mathrm{B}}^{-1/2} = O_{\pm|m} = \frac{1}{2}[(1 \pm \eta_m) I \pm \boldsymbol{r}_m \cdot \boldsymbol{\sigma}]$$

  are known as steering-equivalent observables.
- $O_{+|1}$ and $O_{+|2}$ are coexistent iff[2]

$$(1 - F_1^2 - F_2^2)\left(1 - \frac{\eta_1^2}{F_1^2} - \frac{\eta_2^2}{F_2^2}\right) \leq (\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 - \eta_1 \eta_2)^2,$$

  where

$$F_m = \frac{1}{2}\left(\sqrt{(1+\eta_m)^2 - r_m^2} + \sqrt{(1-\eta_m)^2 - r_m^2}\right).$$

[2] S. Yu, N.-l. Liu, L. Li, and C. H. Oh, PRA 81, 062116 (2010)

Theorem

In the simplest steering scenario, the set of full correlations is EPR nonlocal iff the analog CHSH inequality

$$\begin{aligned}&|\langle(A_1+A_2)B_1\rangle \boldsymbol{b}'_1+\langle(A_1+A_2)B_2\rangle \boldsymbol{b}'_2|\\&+|\langle(A_1-A_2)B_1\rangle \boldsymbol{b}'_1+\langle(A_1-A_2)B_2\rangle \boldsymbol{b}'_2|\leq 2\end{aligned}$$

is violated, where $\boldsymbol{b}'_1, \boldsymbol{b}'_2$ form the dual basis of $\boldsymbol{b}_1, \boldsymbol{b}_2$.

When Bob's measurements are mutually unbiased, the criterion reduces to that derived by Cavalcanti et al.

$$\begin{aligned}&\sqrt{\langle(A_1+A_2)B_1\rangle^2+\langle(A_1+A_2)B_2\rangle^2}\\&+\sqrt{\langle(A_1-A_2)B_1\rangle^2+\langle(A_1-A_2)B_2\rangle^2}\leq 2,\end{aligned}$$

Theorem

The maximal violation S of the analog CHSH inequality by any two-qubit state with correlation matrix T is equal to the maximal violation of the CHSH inequality, namely, $S = 2\sqrt{\lambda_1 + \lambda_2}$*, where* λ_1, λ_2 *are the two largest eigenvalues of* TT^{T}*. Both inequalities are violated iff* $\lambda_1 + \lambda_2 > 1$*.*

Corollary

A two-qubit state can generate Bell-nonlocal correlations in the simplest nontrivial scenario iff it can generate EPR-nonlocal full correlations.

Strict hierarchy between steering and Bell nonlocality

Consider a convex combination of the singlet and a product state,

$$\rho = s(|\Psi_-\rangle\langle\Psi_-|) + (1-s)(|0\rangle\langle 0|) \otimes \frac{I}{2}.$$

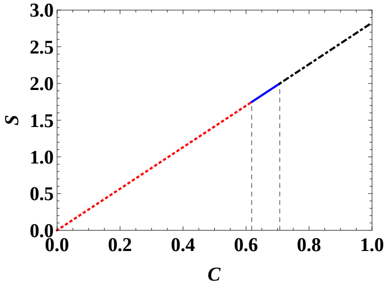

Figure: Black region: states that can generate Bell-nonlocal correlations or EPR-nonlocal full correlations in the simplest scenario. Blue region: states that are steerable in the same scenario, but cannot generate Bell-nonlocal correlations or EPR-nonlocal full correlations.

Simplest and strongest one-way steering

Consider the two-qubit state[3],

$$\rho(p,\theta) = p(|\psi(\theta)\rangle\langle\psi(\theta)|) + (1-p)\Big[\frac{I}{2}\otimes\rho_{\mathrm{B}}(\theta)\Big],$$

where $|\psi(\theta)\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$.

- Violate the analog CHSH inequality iff $p^2\left[1+\sin^2(2\theta)\right] > 1$.
- Not unsteerable from Bob to Alice by arbitrary projective measurements if
$$\cos^2(2\theta) \geq \frac{2p-1}{(2-p)p^3}.$$
- Alice can steer Bob in the simplest scenario iff $p > 1/\sqrt{2}$.

[3] J. Bowles, F. Hirsch, M. T. Quintino, and N. Brunner, PRA 93, 022121 (2016).

Simplest and strongest one-way steering

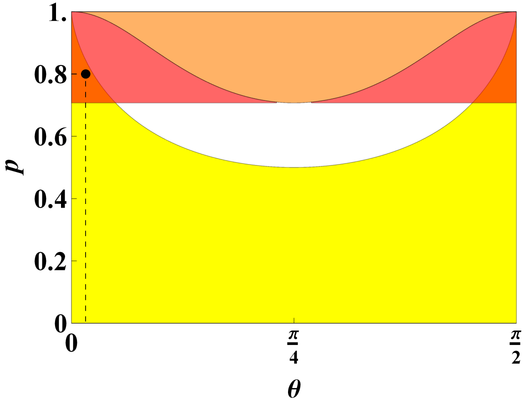

Figure: Orange: violate the (analog) CHSH inequality. Red: steerable from Alice to Bob in the simplest steering scenario, but cannot violate the (analog) CHSH inequality. Yellow: not steerable from Bob to Alice by arbitrary projective measurements. Intersection of the red region and the yellow region: demonstrate the simplest and strongest one-way steering.

Simplest one-way steering with respect to POVMs

Consider the state

$$\rho = \frac{1}{4}\Big[I \otimes I + p\cos(2\theta)\sigma_3 \otimes I + \cos^2\theta I \otimes \sigma_3 + p\cos\theta(\sin\theta\sigma_1 \otimes \sigma_1 - \sin\theta\sigma_2 \otimes \sigma_2 + \cos\theta\sigma_3 \otimes \sigma_3)\Big].$$

- No violation of the (analog) CHSH inequality
- Not unsteerable from Bob to Alice by arbitrary POVMs if

$$\cos^2(2\theta) \geq \frac{2p-1}{(2-p)p^3}.$$

- Alice can steer Bob in the simplest scenario for some parameter range, say $p = 0.825$ and $\theta = 0.020$.

Summary and an open question

- A two-qubit state can generate EPR-nonlocal full correlations in the simplest nontrivial scenario iff it can generate Bell-nonlocal correlations[4].

- When full statistics is taken into account, the same scenario can demonstrate one-way steering and the hierarchy between steering and Bell nonlocality in the simplest and strongest form.

Does there exist a two-qubit state that is not steerable in the simplest scenario, but is steerable in the second simplest scenario in which Alice performs two dichotomic measurements and Bob performs full tomography?

[4]PRA 95, 062111 (2017)

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Thank You!