Exact Uncertainty Relations

Exploring the quantum boundaries

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Outline

Two ways to explore the boundaries

→ From outside (by adding something to QT)

- Nonlocality: local realism is *incompatible* with QT
- Contextuality: noncontextual HV is *incompatible* with QT
- Compatible:
  No-signaling, Information Causality, Local orthogonality, ...

→ From inside (assuming the structure of a Hilbert space)

- Gleason Theorem
- Uncertainty Relations
- Bounds on quantum error-correcting codes
- Universal Cloning Machines
- Quantum metrology
- ...

P.O.S.E of Quantum Theory

- Probability: $P_\Delta = \mathrm{Tr}(\mathcal{E}(\rho) M_\Delta)$
- Observable: POVM $\{M_i \geq 0,\ \sum M_i = I\}$
- State: density matrix $\rho \geq 0$
- Evolution: completely positive map $\rho \mapsto \mathcal{E}(\rho)$

Two kinds of uncertainty relations

- Preparation Uncertainty Relations $\leftarrow \rho \geq 0$
- Measurement Uncertainty Relations $\leftarrow \rho \geq 0$ & $M_I \geq 0$
  - Heisenberg's microscope
  - Joint measurement of two incompatible observables
  - Duality inequality
  - ...

Examples of Uncertainty Relations

Involving only partial information of the statistics such as

- Variance $(\delta_\rho A)^2 = \langle A^2\rangle_\rho - \langle A\rangle_\rho^2$
- Quantum Fisher Information (the convex roof of variance)

$$F_\rho(A) = \sum_i \frac{2(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |\langle i|A|j\rangle|^2 = 4 \min_{\{p_i,|\psi_i\rangle\}} p_i(\delta_{\psi_i} A)^2$$

- Entropy $H(P) = -\sum P_i \ln P_i = -\ln M_0(P)$
- Generalized Entropy

$$M_r(P) = \left(\sum_i P_i^{1+r}\right)^{1/r} \qquad (-1 \leq r \leq \infty)$$

Kennard-Robertson-Schrödinger UR

- Kennard

$$\delta X \delta P \geq \frac{1}{2}$$

- Schrödinger

$$(\delta A)^2 (\delta B)^2 \geq \frac{1}{4}(\langle AB + BA\rangle^2 + \langle AB - BA\rangle^2)$$

- Robertson $|\sigma_X| \geq |i\delta_X|$ where

$$[[\sigma_X]]_{kj} = \frac{1}{2}\langle X_k X_j + X_j X_k\rangle - \langle X_k\rangle\langle X_j\rangle, \quad [[\delta_X]]_{kj} = \frac{i}{2}\langle [X_k, X_j]\rangle$$

Maassen-Uffink UR

$$M_s(P)M_r(Q) \leq c^2$$

- $M_r(P) = \left(\sum_i P_i^{1+r}\right)^{1/r}$
- $r \geq 0, s = -r/(2r+1)$, and $c = \max_{ij} |\langle p_i|q_j\rangle|$

→ $r = s = 0$

$$H(P) + H(Q) \geq -2\ln c$$

→ $r = \infty, s = -1/2$

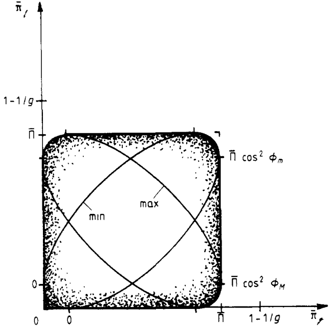
$$\sqrt{Q_{max}} \leq c\sum_n \sqrt{P_n}$$

Larsen's (exact) uncertainty relation

Consider two observables P and Q and take purities

$$M_1(P) = \sum_i P_i^2 = \bar{\pi}_p + \frac{1}{d}, \quad M_1(Q) = \sum_i Q_i^2 = \bar{\pi}_q + \frac{1}{d}$$

as the figures of merit, then it holds



[JPA:Math.Gen 1990]

Formulate the problem

Given a set of observables $\{P, Q, R, \ldots\}$ to determine the exact range $(\langle P\rangle_\rho, \langle Q\rangle_\rho, \langle R\rangle_\rho, \ldots)$ over all possible state ρ.

- Exact UR: constraints on a set of probabilities under which they can be obtained by measuring the given set of observables in certain quantum state.
- Involves the complete statistics obtained by measuring a set of observables;
- Delineates the exact boundary, i.e., whenever the URs are satisfied there is a quantum state in which the measurements of the given set of observables account for the given statistics.

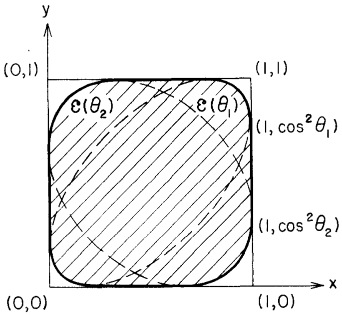
Gleason Theorem

In the case of $d \geq 3$ if all observables represented by complete orthonormal bases $\{\mathcal{O}\}$ are involved then there is essentially no constraints except the trivial one

$$\sum P_i(\mathcal{O}) = 1 \quad (\forall \mathcal{O}).$$

Lenard's exact numerical range

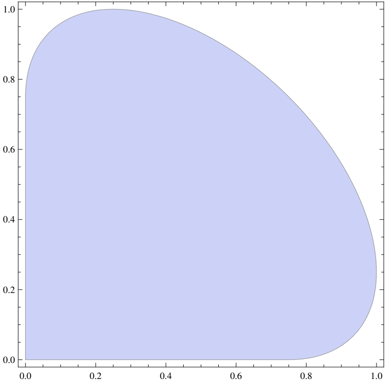
Consider two 2-outcome measurements $\{P, I-P\}$, $\{Q, I-Q\}$ with P, Q being projections (without common eigenvector) then



[J. Function Analysis 1972]

with $x = \langle P \rangle_\rho$, $y = \langle Q \rangle_\rho$, and $\cos^2\theta_{1,2}$ being the largest and smallest eigenvalues of QPQ.

Exact range for two qubit observables



$$P + Q - 2\sqrt{PQ}\cos\frac{\theta}{2} \leq \sin^2\frac{\theta}{2}$$

The boundary

M1: The boundary is the convex hull of possible values attainable by pure states.

M2: Consider the expectations of m observables $\{P_\mu\}_{\mu=1}^{m}$. Let $\mathbf{n} = (n_1, n_2, \ldots, n_m)$ be an arbitrary unit vector and

$$\lambda(\mathbf{n}) = \text{Largest eigenvalue of } \sum_\mu n_\mu P_\mu$$

then the boundary is the hypersurface determined by

$$x_\mu = \langle P_\mu \rangle = \frac{\partial \lambda(\mathbf{n})}{\partial n_\mu}.$$

Main Results: Two Unbiased observables

Consider a d-outcome measurement $\{|n\rangle\langle n|\}_{n=0}^{d-1}$ and a 2-outcome measurement $\{P_\theta = |\theta\rangle\langle\theta|, I - P_\theta\}$ with

$$|\theta\rangle = \frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}|n\rangle.$$

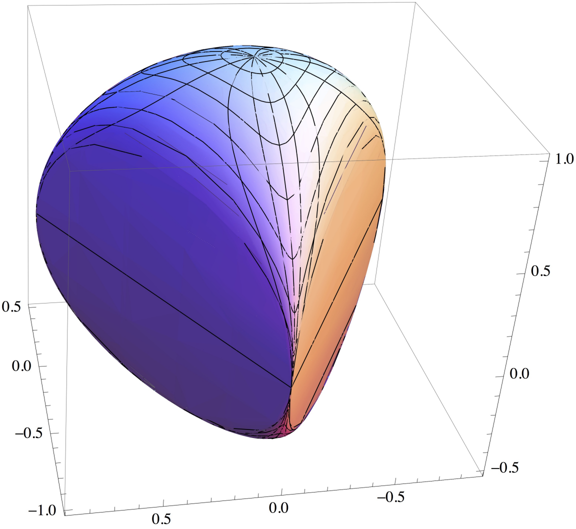
For two probability distributions $\{P_n\}$ and $\{Q, 1-Q\}$ there exists a quantum state ρ such that

$$P_n = \langle n|\rho|n\rangle, \quad Q = \langle P_\theta\rangle,$$

if and only if

$$\max\{0, 2\sqrt{P_{\max}} - \sum_{n=1}^{d-1}\sqrt{P_n}\} \leq \sqrt{dQ} \leq \sum_{n=0}^{d-1}\sqrt{P_n}.$$

d=3



Main Results: Three Unbiased observables

In a 3-level system, consider three 2-outcome measurements $\{P_0 = |0\rangle\langle 0|, I - P_0\}$, $\{Q_0 = |\theta\rangle\langle\theta|, I - Q_0\}$, and $\{R_0 = |\beta\rangle\langle\beta|, I - R_0\}$ with

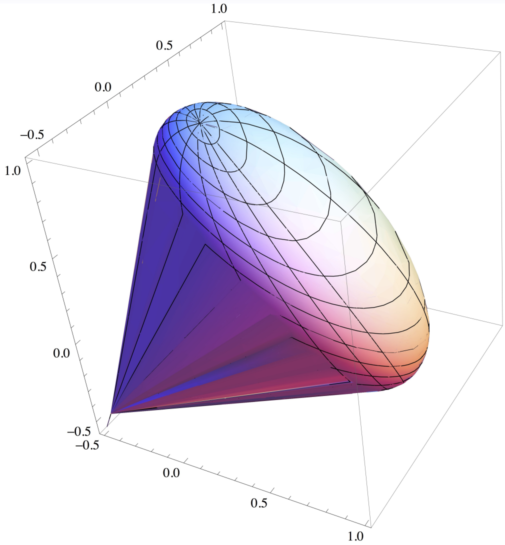
$$|\beta\rangle = \frac{1}{\sqrt{3}}\left(e^{i\frac{2\pi}{3}}|0\rangle + |1\rangle + |2\rangle\right)$$

The values

$$x = \frac{3\langle P_0\rangle_\rho - 1}{2}, \quad y = \frac{3\langle Q_0\rangle_\rho - 1}{2}, \quad z = \frac{3\langle R_0\rangle_\rho - 1}{2}$$

are possible if and only if (x, y, z) belongs to the convex hull of

$$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), \quad E(x^2 + y^2 + z^2 + xy + xz + yz = x + y + z).$$


1.0
0.5
0.0
–0.5
1.0
0.5
0.0
–0.5
–0.5
0.0
0.5
1.0

Main Result: Angular momentum

Consider the measurements of three components $\{J_x, J_y, J_z\}$ of spin-1 system. The corresponding eigenstates are

J_x $\{|x_\pm\rangle = \frac{1}{2}(|-1\rangle \pm \sqrt{2}|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|-1\rangle - |1\rangle)\}$

J_y $\{|y_\pm\rangle = \frac{1}{2}(|-1\rangle \pm i\sqrt{2}|0\rangle - |1\rangle), |+\rangle = \frac{1}{\sqrt{2}}(|-1\rangle + |1\rangle)\}$

J_z $\{|-1\rangle, |0\rangle, |1\rangle\}$

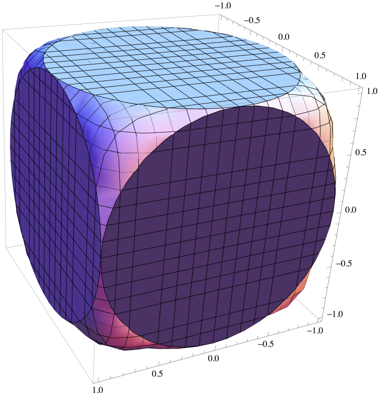
As a result, among three sets of probability distributions $\{P_{x_\pm}, P_-\}, \{P_{y_\pm}, P_+\}, \{P_{\pm 1}, P_0\}$ there are only 5 independent parameters since $P_0 + P_+ + P_- = 1$ and we denote

$$\langle J_x\rangle = \langle P_{x_+} - P_{x_-}\rangle, \langle J_y\rangle = \langle P_{y_+} - P_{y_-}\rangle, \langle J_z\rangle = \langle P_1 - P_{-1}\rangle,$$

$$\theta_x = \langle P_-\rangle, \theta_y = \langle P_+\rangle, \theta_z = \langle P_0\rangle$$

$$\theta_x \langle J_x \rangle^2 + \theta_y \langle J_y \rangle^2 + \theta_z \langle J_z \rangle^2$$

$$\leq 8\theta_x\theta_y\theta_z + \sqrt{(4\theta_y\theta_z - \langle J_x \rangle^2)(4\theta_z\theta_x - \langle J_y \rangle^2)(4\theta_x\theta_y - \langle J_z \rangle^2)}$$

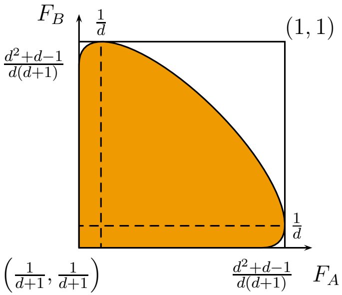


$$x = \frac{\langle J_x \rangle}{2\sqrt{\theta_z\theta_y}},$$

$$y = \frac{\langle J_y \rangle}{2\sqrt{\theta_z\theta_x}},$$

$$z = \frac{\langle J_z \rangle}{2\sqrt{\theta_x\theta_y}}$$

Asymmetric Universal Cloning Machines



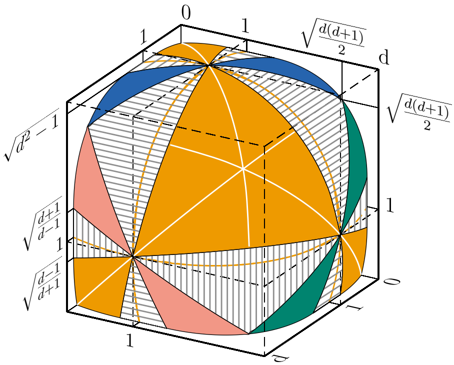
- One to two asymmetric UCM.
- $F = \frac{d+f}{d(d+1)}$
- Exact ranges of

$$f_1 = d\langle \hat{\Phi}_{01} \otimes I_2 \rangle$$
$$f_2 = d\langle \hat{\Phi}_{02} \otimes I_1 \rangle$$

over all states ρ_{012}.

- $|\Phi\rangle_{0k} = \sum_n |n\rangle_0 \otimes |n\rangle_k$

Asymmetric Universal Cloning Machines



$$x = \sqrt{f_1},\ y = \sqrt{f_2},\ z = \sqrt{f_3}$$

- One to three.
- $F = \frac{d+f}{d(d+1)}$
- Exact ranges of

$$f_1 = d\langle \hat{\Phi}_{01} \otimes I_{23} \rangle,$$
$$f_2 = d\langle \hat{\Phi}_{02} \otimes I_{13} \rangle,$$
$$f_3 = d\langle \hat{\Phi}_{03} \otimes I_{12} \rangle,$$

over all states ρ_{0123}.

[MJ&SY, JMP 2010]

Uncertainty relation via parameter estimation

$$\frac{1}{I_A} + \frac{1}{I_B} \geq 3$$

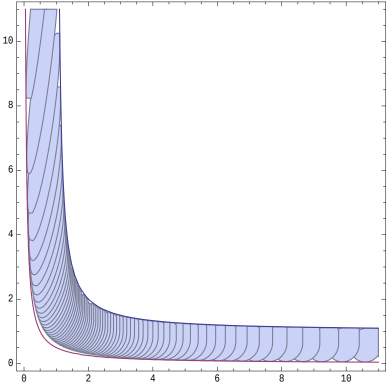
- In order to estimate two parameters $a = \langle A \rangle_\rho$ and $b = \langle B \rangle_\rho$ in a qubit state ρ, we let the qubit interact with a meter qubit

$$U|k\rangle|\phi_0\rangle = |k\rangle|\phi_k\rangle.$$

- Two measurements $A = \vec{a} \cdot \vec{\sigma}$ and $B = \vec{b} \cdot \vec{\sigma}$ are made on the meter and system respectively. The precisions are quantified by the Fisher information of corresponding statistics

$$\Delta\varphi^2 \geq \frac{1}{nI}, \quad I = \sum_k \frac{\dot{p}_k^2}{p_k}.$$

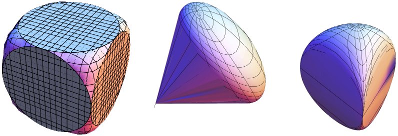
[LS,etal SY,PRA 2017]



$$\frac{1}{I_A}\frac{1}{I_B} \geq (\vec{a} \times \vec{b})^2$$

Summary

- Two kinds of incompatibility lead to two ways of exploring the quantum boundary.
- Two ways to determine the exact range of the statistics, i.e., exact uncertainty relation, of a set of observables, e.g.,



- Two applications illustrated. Might help strengthen the usual uncertainty relations; determine the best performance of some informational operations; establish some measurement uncertainty relations; detection of entanglement (to do).