# INCOMPATIBILITY – a tale in two parts

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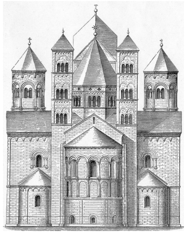


I. Quantum Compression relative to a set of measurements

with L. Rauber, A. Bluhm

II. Information-disturbance tradeoff revisited

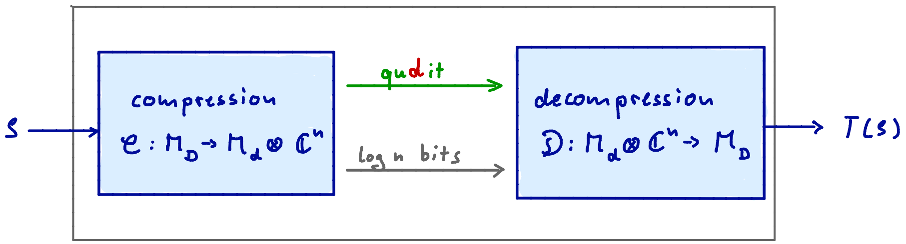
with A. L. Hashagen

# Quantum Compression relative to a set of measurements

**Given:** Set of measurements $\mathcal{S} \subset M_D$

**Aim:** Compress to smaller system s.t. all measurements from $\mathcal{S}$ are preserved.

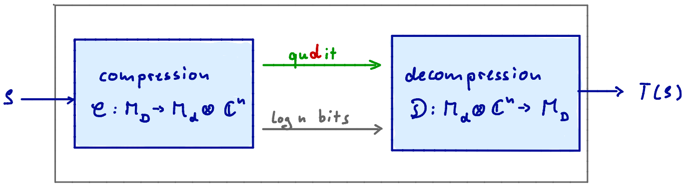
**For free:** Classical info.




$$T := \mathcal{D} \circ \mathcal{E} \; : M_D \to M_D$$

**Def.:** Compression dimension of $\mathcal{S}$ is the smallest $d$ s.t. there is such a map $T$ for which $\forall \varrho \; \forall E \in \mathcal{S}$:

$$\mathrm{tr}[E\varrho] = \mathrm{tr}\left[E\, T(\varrho)\right]$$




$$T := \mathcal{D}\circ\mathcal{E} : M_D \to M_D$$

**Prop.:** [ $4\log_2 D$ bits suffice ]

$$\mathcal{T}_{n,d} := \{ T = \mathcal{D}\circ\mathcal{E} \mid \mathcal{E}: M_D \to M_d\otimes\mathbb{C}^n,\ \mathcal{D}: M_d\otimes\mathbb{C}^n \to M_D \text{ cptp} \}$$

Then $\bigcup_{n\in\mathbb{N}} \mathcal{T}_{n,d} = \mathcal{T}_{m,d}$ with $m = D^4$.

**proof:** ( idea )

$\mathfrak{Z}$ := Choi matrix of $T \in \mathcal{T}_{n,d}$ has Schmidt rank at most $d$

$$\xrightarrow{\text{Caratheodory}} \quad \mathfrak{Z} = \sum_{i=1}^{D^4} \Psi_i \quad \text{with}$$

$$\left.\begin{array}{r} \text{Schmidt-rank}(\Psi_i) \leq d \\ \text{rank}(\Psi_i) = 1 \end{array}\right\} \Rightarrow \Psi_i \mathrel{\hat{=}} \text{element of "}\mathcal{T}_{1,d}\text{"}$$

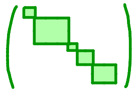
$\square$

**Cor.:** [ Stability ]

For every compact $\mathcal{S} \subset M_D$ there is an $\varepsilon > 0$ s.t. the compression dim. does not change if we allow for

$$\big| \operatorname{tr}[E\rho] - \operatorname{tr}[E\,T(\rho)] \big| \leq \varepsilon .$$

# Operator algebraic bounds on $d$

$C^*(\mathcal{S}) :=$ $C^*$-algebra generated by $\mathcal{S}$ $\quad \simeq$ 

**Thm.:** $\boxed{\text{If } C^*(\mathcal{S}) \simeq \bigoplus_{i=1}^{b} M_{D_i}, \text{ then } \min_i \{D_i\} \leq d \leq \max_i \{D_i\}}$

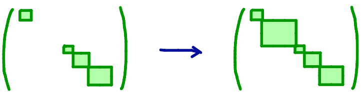
Proof exploits a result from Arveson '72 ...

**Remarks:**

- Bounds are essentially tight in terms of $C^*(\mathcal{S})$.
- If $\mathcal{S}$ contains effects of two binary vN measurements, then $D_i \leq 2$.
- The set of hermitian pairs $(E_1, E_2) \in M_D \times M_D$ s.t. $C^*(\{E_1, E_2\}) \neq M_D$ has measure zero.

**Thm.:** $\boxed{d \in \{D_i\} \text{ can be computed via a SDP}}$

main ingredient: check whether there is a unital cp map s.t.



**Cor.:** $\boxed{\text{Compression to dim } d \text{ is feasible with } n \leq b.}$

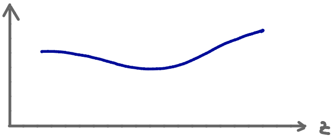
# Complex analytic bound

**Thm.:** Let $E_1, E_2 \in \mathcal{S}$. The smallest among the degrees of the irred. factors of

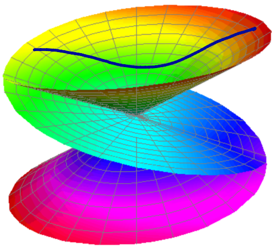
$$p(\lambda, z) := \det(\lambda \mathbb{1} - E_1 - z E_2)$$

is a lower bound on $d$.

**proof** (idea): $\| E_1 + z E_2 \|_\infty =: f(z)$




$\downarrow$ analytic continuation



'k-valued' Riemann surface where $k$ = degree of an irred. factor of $p$

If there is a compression to dim. $d$, there are $F_1, F_2 \in M_d$ s.t. $f(z) = \| F_1 + z F_2 \|_\infty$. Hence, $k \leq d$. $\square$

**Note:** This bound still holds for multiple copies $\mathcal{S}^{\otimes m}$ & positive maps $\mathcal{D}, \mathcal{E}$.

# Open problems

- $D = \infty$

- Efficient algorithm for $(\varepsilon, d)$-tradeoff

- Variations of the problem:

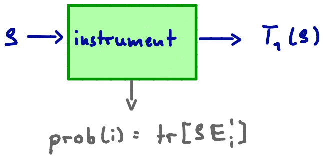
    - restrict set of states & observables
    - allow to use different observables
      (or prepare different states) $\longrightarrow$ PSD rank

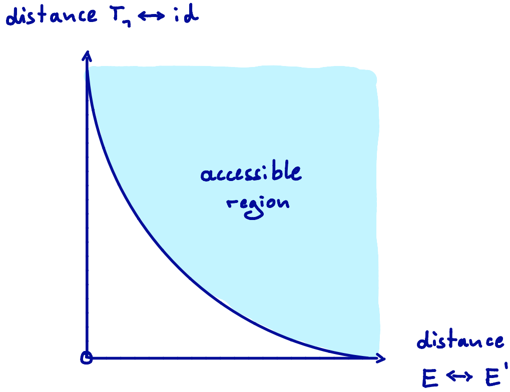
$$\boxed{\text{Information - disturbance tradeoff}}$$

Incompatible tasks:

- measuring a POVM $(E_1, \ldots, E_m)$
- not disturbing the system

What's the optimal tradeoff?




Examples for distance measures: $\Delta(T_1) := \| T_1 - id \|_\diamond$

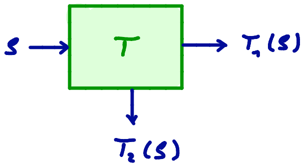
$\delta(E') := \max_i \| E_i - E_i' \|_\infty$

Prop.: In this case the accessible region is the feasible set of an SDP.

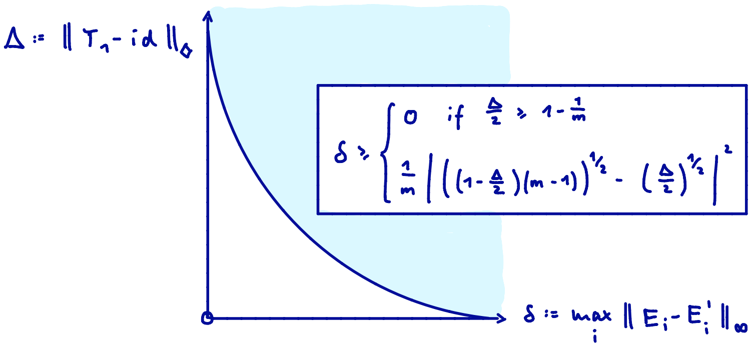
proof: Scheiderer '12: $S \subseteq \mathbb{R}^2$ convex semialgebraic

$\Leftrightarrow$ $S$ is feasible set of an SDP. $\square$

# Sharp measurements




$$E_i' = T_2^*(E_i)$$




→ independent of dimension

## More general distance measures :

$\delta(E')$, $\Delta(T_1)$ in $\mathbb{R}_+$, zero in ideal case and

(i) convex

(ii) 'basis independent' • $\Delta\left(u T_1 (u^* \cdot u) u^*\right) = \Delta(T_1)$ $\quad \forall u \in \mathcal{U}(d)$

• $\delta\left( \left( u_{\pi(i)} E'_{\pi^{-1}(i)} u^*_{\pi(i)} \right)_{i=1}^m \right) = \delta(E')$ $\quad \forall \pi \in S_m$

(iii) 'ess. diagonal' • $\delta\left( \left( u E_i' u^* \right)_{i=1}^m \right) = \delta(E')$ $\quad \forall$ diag. $u \in \mathcal{U}(d)$

(true if $\delta$ comes from 'worst-case' or 'average-case' w.r.t. $S$)

# Sharp measurements - symmetry reduction

**Lemma:** [à la Werner]
w.l.o.g. $T = (u \otimes u) T (u^* \cdot u) (u \otimes u)^* \quad \forall u \in G$
($G$ generated by permutations & diagonal unitaries)

$\Rightarrow \quad T_j = \alpha_j \, tr[\cdot] \frac{\mathbb{1}}{d} + \beta_j \, id + \gamma_j \sum_i |i\rangle\langle i| \langle i|\cdot|i\rangle$

$\Rightarrow \quad \delta$ is monotone function of $\alpha_2$

Set of symmetric $T$ has dim. 12.

Relevant Choi matrices are 'contraction tensors':

 $:= \sum_{ij} |ii\rangle\langle jj| \otimes \mathbb{1}_d$

 $:= \mathbb{1}_{d^3}$

 $:= \sum_i |ii\rangle\langle ii| \otimes \mathbb{1}_d$

 $:= \mathbb{1}_d \otimes \sum_i |ii\rangle\langle ii|$

generate algebra $\mathcal{A}$:

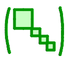
 $=:$ 

 $=:$ 

 $=:$ 

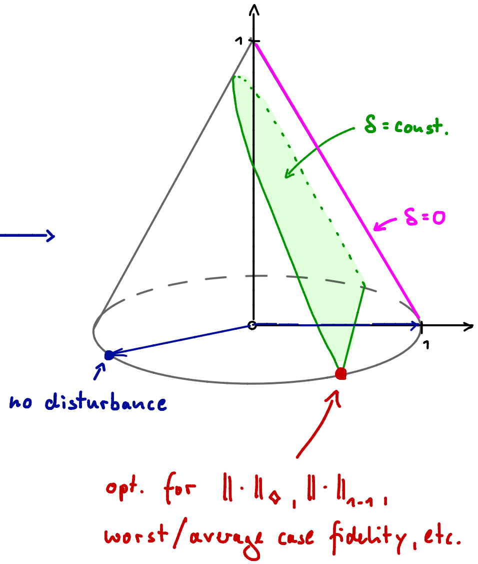
no more elements

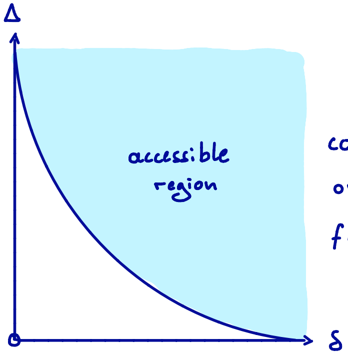
$\Rightarrow \quad \dim(\mathcal{A}) = 7$, $\mathcal{A}$ non-com. $\quad \Rightarrow \quad \mathcal{A} \simeq M_2 \oplus \mathbb{C}^3$



$$\begin{tikzcd} T : M_d \to M_d \otimes M_d \arrow[d, "\text{Choi-Jamiołkowski}"] \\ M_{d^3} \arrow[d, "\text{sym.}"] \\ \mathcal{A} \arrow[r, "C^*\text{-iso.}"] & M_2 \oplus \mathbb{C}^3 \\ & \cup \\ & \text{states} \end{tikzcd}$$

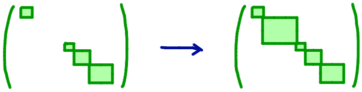
$$\boxed{\begin{gathered} T = \sum_i \Big[ P_i \cdot P_i + \lambda \langle i|\cdot|i\rangle \big( \mathbb{1} - |i\rangle\langle i| \big) \Big] \otimes |i\rangle\langle i| \\ P_i := \mu \mathbb{1} + \nu |i\rangle\langle i| \,, \quad \mu, \nu, \lambda \in \mathbb{R} \end{gathered}}$$

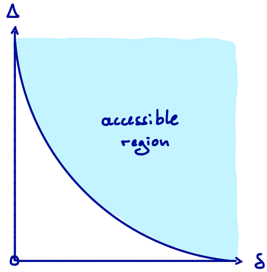


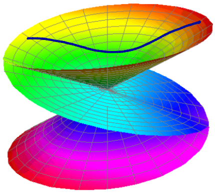



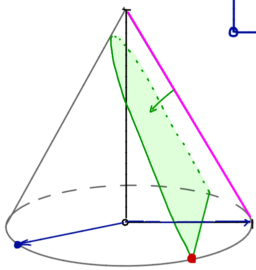

computable by going over a 2-dim. set for all Δ, δ satisfying (i) - (iii)

# Summary







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