Entropic uncertainty relations - The measurement case

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Preparation uncertainties

A, B = two observables (POVMs) with outcomes Ω_{A} and Ω_{B}

For any state ρ,

$$\mathsf{A}^{\rho}(X) = \mathrm{tr}\,[\rho\mathsf{A}(X)] \qquad \mathsf{B}^{\rho}(Y) = \mathrm{tr}\,[\rho\mathsf{B}(Y)] \qquad X \subset \Omega_{\mathsf{A}},\ Y \subset \Omega_{\mathsf{B}}$$

PUR = any constraint relating all the probabilities A^{ρ} and B^{ρ} evaluated at the same state ρ (or, tipically, their spreads); no joint measurement or approximate joint measurement of A and B is involved.

Preparation uncertainties

Robertson-Schrödinger uncertainty relations for two selfadjoint operators (1929-30)

$$\mathbb{V}_\rho(A)\mathbb{V}_\rho(B) \geq \left|\frac{1}{2}\mathbb{E}_\rho(\{A,B\}) - \mathbb{E}_\rho(A)\mathbb{E}_\rho(B)\right|^2 + \frac{1}{4}|\mathbb{E}_\rho([A,B])|^2$$
$$\geq \frac{1}{4}|\mathbb{E}_\rho([A,B])|^2$$

where

$$\mathbb{E}_\rho(X) = \mathrm{tr}\,[\rho X] \qquad \mathbb{V}_\rho(X) = \mathbb{E}_\rho(X^2) - \mathbb{E}_\rho(X)^2$$

For position and momentum

$$\mathbb{V}_\rho(Q)\mathbb{V}_\rho(P) \geq \frac{\hbar^2}{4}$$

Preparation uncertainties

Maassen-Uffnik entropic uncertainty relations for two sharp observables (PVMs) in finite dimension (1988)

$$H(\mathsf{A}^\rho) + H(\mathsf{B}^\rho) \geq -2 \log \max_{x,y} |\langle\, a_x \,|\, b_y \,\rangle|$$

where

$$\mathsf{A}(x) = |a_x\rangle\langle a_x| \qquad\qquad \mathsf{B}(y) = |b_y\rangle\langle b_y|$$

and H is the *Shannon entropy*

$$H(p) = -\sum_z p(z) \log p(z) \quad (\text{with } 0 \log 0 \equiv 0)$$

Preparation uncertainties

Krishna-Parthasarathy entropic uncertainty relations for two generic observables (POVMs) in finite dimension (2002)

$$H(\mathsf{A}^\rho) + H(\mathsf{B}^\rho) \geq -2 \log \max_{x,y} \left\| \mathsf{A}(x)^{1/2} \mathsf{B}(y)^{1/2} \right\|$$

For position and momentum (with $\hbar = 1$)

$H(\mathsf{Q}^\rho) + H(\mathsf{P}^\rho) > 0$ (Hirschman 1957)

$H(\mathsf{Q}^\rho) + H(\mathsf{P}^\rho) \geq \log(\pi e)$ (Beckner & Bialynicki-Birula, Mycielski 1975)

Measurement uncertainties

A, B = two *target observables* (POVMs) with outcomes Ω_A and Ω_B

M = a bi-observables, i.e., a POVM with outcomes $\Omega_A \times \Omega_B$

$M_{[1]}$, $M_{[2]}$ = the two marginals of M

We regard M as an *approximate joint measurement* of A and B

MUR = a lower bound for the “errors” of any approximate joint measurement M of A and B

MURs require to fix an *error function* describing how well $M_{[1]}^{\rho}$ and $M_{[2]}^{\rho}$ approximate the target distributions A^{ρ} and B^{ρ}, for any state ρ.

Measurement uncertainties

The *relative entropy* of a probability p w.r.t. a probability q (or Kullback-Leibler divergence of q from p) is

$$S(p\|q) = \begin{cases} \sum_{x \in \mathrm{supp}\, p} p(x) \log \frac{p(x)}{q(x)} & \text{if } \mathrm{supp}\, p \subseteq \mathrm{supp}\, q, \\ +\infty & \text{otherwise.} \end{cases}$$

Properties:

- $S(p\|q) \geq 0$, and $S(p\|q) = 0$ iff $p = q$;
- $S(\,\cdot\,\|\,\cdot\,)$ is jointly convex and LSC;
- $S(p\|q)$ is independent of the labeling of the outcomes.

Measurement uncertainties

The total error in approximating (A, B) with the bi-observable M is

$$S\big(\mathsf{A}^\rho \| \mathsf{M}^\rho_{[1]}\big) + S\big(\mathsf{B}^\rho \| \mathsf{M}^\rho_{[2]}\big) \qquad \text{in the state } \rho$$

Taking the worst possible case w.r.t. ρ, we get the *entropic divergence* of M from (A, B):

$$D\big(\mathsf{A}, \mathsf{B} \| \mathsf{M}\big) = \sup_\rho \Big\{ S\big(\mathsf{A}^\rho \| \mathsf{M}^\rho_{[1]}\big) + S\big(\mathsf{B}^\rho \| \mathsf{M}^\rho_{[2]}\big) \Big\}$$

The *entropic incompatibility degree* of A and B is

$$c_{\rm inc}(\mathsf{A}, \mathsf{B}) = \inf_{\mathsf{M}} D\big(\mathsf{A}, \mathsf{B} \| \mathsf{M}\big)$$

Entropic measurement uncertainty relations:

$$\forall \text{ bi-observable } \mathsf{M} \ \ \exists \rho \ \text{ s.t. } \ S\big(\mathsf{A}^\rho \| \mathsf{M}^\rho_{[1]}\big) + S\big(\mathsf{B}^\rho \| \mathsf{M}^\rho_{[2]}\big) \geq c_{\rm inc}(\mathsf{A}, \mathsf{B})$$

Measurement uncertainties

Relation with preparation uncertainty:

$$c_{\text{inc}}(\mathsf{A},\mathsf{B}) + c_{\text{prep}}(\mathsf{A},\mathsf{B}) \leq \log|\Omega_{\mathsf{A}}| + \log|\Omega_{\mathsf{B}}|$$

where

$$c_{\text{prep}}(\mathsf{A},\mathsf{B}) = \inf_{\rho}[H(\mathsf{A}^{\rho}) + H(\mathsf{B}^{\rho})]$$

N. B.: $c_{\text{prep}}(\mathsf{A},\mathsf{B})$ can be $\neq 0$ even if A and B are compatible!

Main results

Theorem (Properties of c_{inc})

(i) $c_{\mathrm{inc}}(\mathsf{A}, \mathsf{B}) = c_{\mathrm{inc}}(\mathsf{B}, \mathsf{A})$

(ii) *For all observables* A, B

$$0 \leq c_{\mathrm{inc}}(\mathsf{A}, \mathsf{B}) \leq \log \frac{2(d+1)}{d + 2 + d \min_x \mathsf{A}^{\rho_0}(x)} + \log \frac{2(d+1)}{d + 2 + d \min_y \mathsf{B}^{\rho_0}(y)} \leq 2 \log 2$$

where $\rho_0 = \mathbb{1}/d$ *is the maximally mixed state*

(iii) *The set of* optimal approximate joint measurements

$$\mathcal{M}_{\mathrm{inc}}(\mathsf{A}, \mathsf{B}) = \arg\min_{\mathsf{M}} D\big(\mathsf{A}, \mathsf{B} \| \mathsf{M}\big)$$

is nonempty, convex and compact

Main results

Theorem (Properties of c_{inc})

(iv) $c_{\mathrm{inc}}(\mathsf{A},\mathsf{B}) = 0$ *if and only if* A *and* B *are compatible, and in this case* $\mathcal{M}_{\mathrm{inc}}(\mathsf{A},\mathsf{B})$ *is the set of all the joint measurements of* A *and* B*.*

(v) *If a group G acts*

- on $\Omega_{\mathsf{A}} \times \Omega_{\mathsf{B}}$ *(by means of a suitable action)*
- on $\mathcal{H}$ *(by means of a projective unitary representation)*

and A *and* B *are covariant, then there is always a G-covariant element in* $\mathcal{M}_{\mathrm{inc}}(\mathsf{A},\mathsf{B})$

Examples

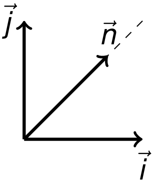
For two orthogonal sharp spin-1/2 observables

$$\mathsf{X}(x) = \frac{1}{2}(\mathbb{1} + x\sigma_1) \qquad \mathsf{Y}(y) = \frac{1}{2}(\mathbb{1} + y\sigma_2) \qquad (x, y = \pm 1)$$

we have

$$c_{\rm inc}(\mathsf{X}, \mathsf{Y}) = \log \frac{2\sqrt{2}}{\sqrt{2}+1}$$

$$\mathcal{M}_{\rm inc}(\mathsf{X}, \mathsf{Y}) = \{\mathsf{M}_0\} \quad \text{with} \quad \mathsf{M}_0(x, y) = \frac{1}{4}\left[\mathbb{1} + \frac{x}{\sqrt{2}}\sigma_1 + \frac{y}{\sqrt{2}}\sigma_2\right]$$



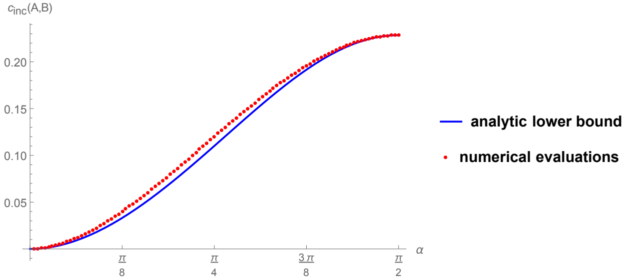
The symmetry group is the dihedral group D_4 generated by the 180° rotations along $\vec{i}$ and $\vec{n}$.

Examples

For two nonorthogonal sharp spin-1/2 observables

$$\mathsf{A}(x) = \frac{1}{2}(\mathbb{1} + x\,\mathbf{a}\cdot\boldsymbol{\sigma}) \qquad \mathsf{B}(y) = \frac{1}{2}(\mathbb{1} + y\,\mathbf{b}\cdot\boldsymbol{\sigma}) \qquad \mathbf{a}\cdot\mathbf{b} = \cos\alpha$$

an analytic lower bound can be given for $c_{\mathrm{inc}}(\mathsf{A},\mathsf{B})$.



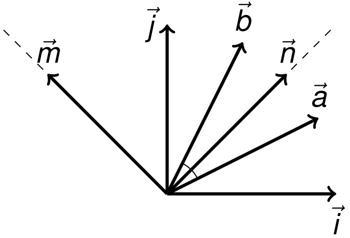
Examples

For two nonorthogonal sharp spin-$1/2$ observables

$$\mathsf{A}(x) = \frac{1}{2}(\mathbb{1} + x\,\mathbf{a}\cdot\boldsymbol{\sigma}) \qquad \mathsf{B}(y) = \frac{1}{2}(\mathbb{1} + y\,\mathbf{b}\cdot\boldsymbol{\sigma}) \qquad \mathbf{a}\cdot\mathbf{b} = \cos\alpha$$

an analytic lower bound can be given for $c_{\mathrm{inc}}(\mathsf{A},\mathsf{B})$.

Contrary to the orthogonal case, the (unique) covariant $\mathsf{M}_0 \in \mathcal{M}_{\mathrm{inc}}(\mathsf{A},\mathsf{B})$ does not have noisy versions of A and B as marginals.



The symmetry group is the dihedral group D_2 generated by the 180° rotations along $\vec{n}$ and $\vec{m}$.

Examples

For two Fourier-related MUBs in prime-power dimension $d = p^n$

$$\mathsf{Q}(x) = |e_x\rangle\langle e_x| \qquad \mathsf{P}(y) = F^{-1}\mathsf{Q}(y)F \qquad (x, y \in \mathbb{F}_d)$$

$$F_{x,y} = \frac{1}{\sqrt{d}} \exp\left(-\frac{2\pi i}{p}\mathrm{Tr}\,(xy)\right)$$

we have

$$c_{\mathrm{inc}}(\mathsf{A}, \mathsf{B}) \geq \log \frac{2\sqrt{d}}{\sqrt{d}+1}$$

and, if p is odd,

$$\mathcal{M}_{\mathrm{inc}}(\mathsf{Q}, \mathsf{P}) = \{\mathsf{M}_0\} \quad \text{with} \quad \mathsf{M}_0(x, y) = \frac{1}{2(d+\sqrt{d})}|\psi_{x,y}\rangle\langle\psi_{x,y}|$$

where

$$\psi_{x,y} = e_x + \exp\left(-\frac{2\pi i}{p}\mathrm{Tr}\,(xy)\right) Fe_{-y}$$

The symmetry group is the translation group of the finite phase-space $\mathbb{F}_d^2$

Error / disturbance

Definition (Heinosaari, Wolf (2010))

The observable A *can be measured without disturbing* B if there exists an instrument $\mathcal{J}$ on Ω_{A} such that (in the Heisenberg picture)

$$\mathcal{J}_x(\mathbb{1}) = \mathsf{A}(x) \qquad \forall x \in \Omega_{\mathsf{A}}$$
$$\mathcal{J}_{\Omega_{\mathsf{A}}}(\mathsf{B}(y)) = \mathsf{B}(y) \qquad \forall y \in \Omega_{\mathsf{B}}$$

In this case, the bi-observable

$$[\mathcal{J}(B)](x,y) := \mathcal{J}_x(\mathsf{B}(y))$$

is a joint measurement of A and B

Error / disturbance

Sequential measurements of approximate versions of A followed by B:

$$\mathcal{M}(\Omega_{\mathsf{A}}; \mathsf{B}) = \{\mathcal{J}(B) \mid \mathcal{J} \text{ is an instrument on } \Omega_{\mathsf{A}}\}$$

If $\mathcal{J}(B) \in \mathcal{M}(\Omega_{\mathsf{A}}; \mathsf{B})$, then in general

$$\mathcal{J}(B)_{[1]} = \mathcal{J}_{\cdot}(\mathbb{1}) \neq \mathsf{A} \qquad (\mathcal{J} \textit{ approximates } \mathsf{A})$$

$$\mathcal{J}(B)_{[2]} = \mathcal{J}_{\Omega_{\mathsf{A}}}(\mathsf{B}(\cdot)) \neq \mathsf{B} \qquad (\mathcal{J} \textit{ disturbs } \mathsf{B})$$

Entropic error / disturbance coefficient

We can define the *entropic error/disturbance coefficient*

$$c_{\mathrm{ed}}(\mathsf{A};\mathsf{B}) = \inf_{\mathsf{M}\in\mathcal{M}(\Omega_\mathsf{A};\mathsf{B})} D\big(\mathsf{A},\mathsf{B}\|\mathsf{M}\big)$$

$$= \inf_{\mathsf{M}\in\mathcal{M}(\Omega_\mathsf{A};\mathsf{B})} \sup_{\rho} \Big\{ S\big(\mathsf{A}^\rho\|\mathsf{M}^\rho_{[1]}\big) + S\big(\mathsf{B}^\rho\|\mathsf{M}^\rho_{[2]}\big) \Big\}$$

Note that c_{ed} is NOT symmetric

Entropic error/disturbance uncertainty relations:

$$\forall \text{ instrument } \mathcal{J} \;\; \exists\rho \;\text{ s. t. }\; S\big(\mathsf{A}^\rho\|\mathcal{J}(\mathsf{B})^\rho_{[1]}\big) + S\big(\mathsf{B}^\rho\|\mathcal{J}(\mathsf{B})^\rho_{[2]}\big) \geq c_{\mathrm{ed}}(\mathsf{A};\mathsf{B})$$

Main results

Theorem (Properties of c_{ed})

(i) $c_{\mathrm{inc}}(\mathsf{A},\mathsf{B}) \leq c_{\mathrm{ed}}(\mathsf{A};\mathsf{B})$

(ii) $c_{\mathrm{inc}}(\mathsf{A},\mathsf{B}) = c_{\mathrm{ed}}(\mathsf{A};\mathsf{B})$ *if* B *is sharp*

(iii) *The same bounds of* c_{inc} *hold for* c_{ed}

(iv) *The set of* optimal approximate measurements of A resulting in the minimal disturbance on B

$$\mathcal{M}_{\mathrm{ed}}(\mathsf{A};\mathsf{B}) = \underset{\mathsf{M}\in\mathcal{M}(\Omega_{\mathsf{A}};\mathsf{B})}{\arg\min} \; D\big(\mathsf{A},\mathsf{B}\|\mathsf{M}\big)$$

is nonempty, convex and compact

(v) $c_{\mathrm{ed}}(\mathsf{A};\mathsf{B}) = 0$ *if and only if* A *can be measured without disturbing* B*, and in this case* $\mathcal{M}_{\mathrm{ed}}(\mathsf{A};\mathsf{B})$ *is the set of all the sequential measurements of* A *followed by* B *which do not disturb* B

Generalization to $n > 2$ observables

The incompatibility index c_{inc} can be easily generalized to the case of n observables $\mathsf{A}_1, \ldots, \mathsf{A}_n$:

$$c_{\mathrm{inc}}(\mathsf{A}_1, \ldots, \mathsf{A}_n) = \inf_{\mathsf{M}} \sup_{\rho} \sum_{i=1}^{n} S\big(\mathsf{A}_i^{\rho} \| \mathsf{M}_{[i]}^{\rho}\big)$$

$$\mathcal{M}_{\mathrm{inc}}(\mathsf{A}_1, \ldots, \mathsf{A}_n) = \arg\min_{\mathsf{M}} \sup_{\rho} \sum_{i=1}^{n} S\big(\mathsf{A}_i^{\rho} \| \mathsf{M}_{[i]}^{\rho}\big)$$

For example, for three orthogonal sharp spin-$1/2$ observables,

$$c_{\mathrm{inc}}(\mathsf{X}, \mathsf{Y}, \mathsf{Z}) = \log \frac{2\sqrt{3}}{\sqrt{3} + 1}$$

The Q-P case

For the position Q and momentum P on $\mathbb{R}$, and any bi-observable M on $\mathbb{R}^2$, we can still define the entropic divergence

$$D(\mathsf{Q},\mathsf{P}\|\mathsf{M}) = \sup_{\rho}\left\{S(\mathsf{Q}^{\rho}\|\mathsf{M}^{\rho}_{[1]}) + S(\mathsf{P}^{\rho}\|\mathsf{M}^{\rho}_{[2]})\right\}$$

where the relative entropy of a probability measure μ w.r.t. a probability measure ν is

$$S(\mu\|\nu) = \begin{cases} \int \left(\log \frac{\mathrm{d}\mu(x)}{\mathrm{d}\nu(x)}\right) \mathrm{d}\mu(x) & \text{if } \mu \text{ has density w.r.t. } \nu \\ +\infty & \text{otherwise} \end{cases}$$

However, the Radon-Nikodym derivatives $\frac{\mathrm{d}\mathsf{Q}^{\rho}(x)}{\mathrm{d}\mathsf{M}^{\rho}_{[1]}(x)}$ and $\frac{\mathrm{d}\mathsf{P}^{\rho}(y)}{\mathrm{d}\mathsf{M}^{\rho}_{[2]}(y)}$ may be difficult to evaluate.

The *Q*-*P* case

For this reason, we restrict to a very particular case:

- Gaussian states;
- Gaussian and covariant approximating bi-observables.

Covariance is understood w.r.t. the phase-space translation group:

$$\mathsf{M}(Z + (x, y)) = W(x, y)\mathsf{M}(Z)W(x, y)^* \qquad \forall Z \in \mathcal{B}(\mathbb{R}^2),\ (x, y) \in \mathbb{R}^2$$

where the $W(x, y)$'s are the *Weyl operators*

$$W(x, y) = \exp\left[\frac{\mathrm{i}}{\hbar}(yQ - xP)\right]$$

The Q-P case

With these assumptions,

$$\mathsf{Q}^{\rho} = N(\mathbb{E}_{\rho}(Q), \mathbb{V}_{\rho}(Q)) \qquad \mathsf{P}^{\rho} = N(\mathbb{E}_{\rho}(P), \mathbb{V}_{\rho}(P))$$

$$\mathsf{M}^{\rho}_{[1]} = \mathsf{Q}^{\rho} * N(\mathbb{E}_{\mathsf{M}}(Q), \mathbb{V}_{\mathsf{M}}(Q)) \qquad \mathsf{M}^{\rho}_{[2]} = \mathsf{P}^{\rho} * N(\mathbb{E}_{\mathsf{M}}(P), \mathbb{V}_{\mathsf{M}}(P))$$

where $\mathbb{E}_{\mathsf{M}}(Q), \mathbb{E}_{\mathsf{M}}(P) \in \mathbb{R}$, $\mathbb{V}_{\mathsf{M}}(Q) > 0$, $\mathbb{V}_{\mathsf{M}}(P) > 0$, and

$$\mathbb{V}_{\mathsf{M}}(Q)\mathbb{V}_{\mathsf{M}}(P) \geq \frac{\hbar^2}{4}$$

However,

$$D^{\mathcal{G}}(\mathsf{Q}, \mathsf{P} \| \mathsf{M}) := \sup_{\rho \text{ Gaussian}} \left\{ S(\mathsf{Q}^{\rho} \| \mathsf{M}^{\rho}_{[1]}) + S(\mathsf{P}^{\rho} \| \mathsf{M}^{\rho}_{[2]}) \right\} = +\infty$$

because e.g. $S(\mathsf{Q}^{\rho} \| \mathsf{M}^{\rho}_{[1]}) \to +\infty$ when $\dfrac{\mathbb{V}_{\rho}(Q)}{\mathbb{V}_{\mathsf{M}}(Q)} \to 0$

This is a classical effect!

The Q-P case

In order to avoid classical effects, we fix a threshold $\epsilon > 0$

$$\sup_{\substack{\rho \text{ Gaussian} \\ \mathbb{V}_\rho(Q)\geq\epsilon}} S(\mathrm{Q}^\rho \| \mathrm{M}^\rho_{[1]}) = \frac{\log e}{2}\left[\ln\left(1+\frac{\mathbb{V}_\mathrm{M}(Q)}{\epsilon}\right)+\frac{\mathbb{E}_\mathrm{M}(Q)^2-\mathbb{V}_\mathrm{M}(Q)}{\mathbb{V}_\mathrm{M}(Q)+\epsilon}\right]$$

$$\inf_{\substack{\mathrm{M}\text{ Gaussian} \\ \text{and covariant}}} \sup_{\substack{\rho \text{ Gaussian} \\ \mathbb{V}_\rho(Q)\geq\epsilon}} S(\mathrm{Q}^\rho \| \mathrm{M}^\rho_{[1]}) = 0$$

If there were not quantum effects, for any $\epsilon_1, \epsilon_2 > 0$,

$$c_{\mathrm{inc}}(\mathrm{Q},\mathrm{P};\epsilon) := \inf_{\substack{\mathrm{M}\text{ Gaussian} \\ \text{and covariant}}} \sup_{\substack{\rho \text{ Gaussian} \\ \mathbb{V}_\rho(Q)\geq\epsilon_1 \\ \mathbb{V}_\rho(P)\geq\epsilon_2}} \left\{ S(\mathrm{Q}^\rho \| \mathrm{M}^\rho_{[1]}) + S(\mathrm{P}^\rho \| \mathrm{M}^\rho_{[2]}) \right\} = 0$$

The Q-P case

But since there are quantum effects,

$$c_{\mathrm{inc}}(\mathsf{Q},\mathsf{P};\epsilon)\begin{cases} = (\log \mathrm{e}) \left\{ \ln\left(1+\frac{\hbar}{2\sqrt{\epsilon_1\epsilon_2}}\right) - \frac{\hbar}{\hbar+2\sqrt{\epsilon_1\epsilon_2}} \right\} & \text{if } \epsilon_1\epsilon_2 \geq \frac{\hbar^2}{4} \\ \geq (\log \mathrm{e}) \left(\ln 2 - \frac{1}{2}\right) & \text{if } \epsilon_1\epsilon_2 < \frac{\hbar^2}{4} \end{cases}$$

Moreover, when $\epsilon_1\epsilon_2 \geq \dfrac{\hbar^2}{4}$, the optimal Gaussian and covariant bi-observable is unique.

For every Gaussian and covariant bi-observable M, the total information loss $S\big(\mathsf{Q}^\rho \| \mathsf{M}^\rho_{[1]}\big) + S\big(\mathsf{P}^\rho \| \mathsf{M}^\rho_{[2]}\big)$ can exceed the lower bound $c_{\mathrm{inc}}(\mathsf{Q},\mathsf{P};\epsilon)$ even if we forbid states with too peaked target distributions.

The Q-P case

Further generalizations:

- vector valued position and momentum $\vec{Q}$, $\vec{P}$:
  $c_{\rm inc}(\mathsf{Q}, \mathsf{P}; \epsilon)$ linearly scales with the dimension n
- position and momentum along different directions $\vec{a} \cdot \vec{Q}$, $\vec{b} \cdot \vec{P}$:
  $c_{\rm inc}(\mathsf{Q}, \mathsf{P}; \epsilon) \to 0$ with continuity as $\vec{a} \cdot \vec{b} \to 0$.

References

- A. Barchielli, M. Gregoratti and A. Toigo, *Measurement uncertainty relations for discrete observables: Relative entropy formulation*, coming (hopefully) soon (2017)
- A. Barchielli, M. Gregoratti and A. Toigo, *Measurement uncertainty relations for position and momentum: Relative entropy formulation*, Entropy **19**, 301 (2017)