Measurement uncertainty relations for finite observables

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September 6, 2017

Observables A and B

$$A : \{(x, P(x))\} \quad x \in \mathcal{A}$$

$$B : \{(x', Q(x'))\} \quad x' \in \mathcal{B}$$

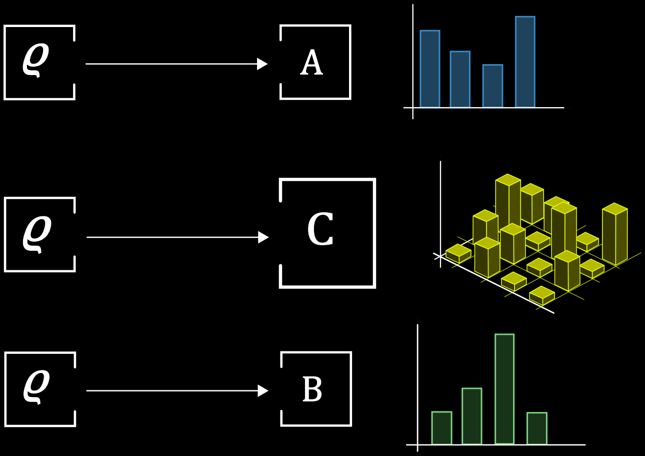
If there is no perfect joint measurement:

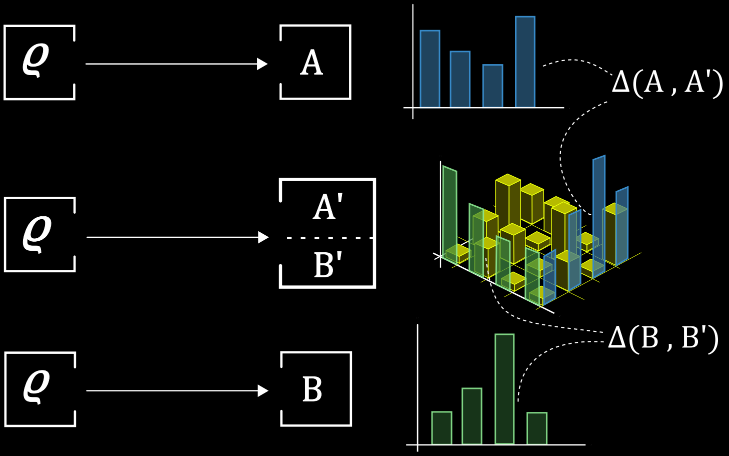
Can we find an approximate joint measurement?

What is a good approximation?

Can we compute it?

$$C : \{(z, R(z))\} \quad z \in \mathcal{A} \times \mathcal{B}$$


ϱ
A
ϱ
C
ϱ
B


ϱ
A
ϱ
A'
B'
ϱ
B
Δ(A , A')
Δ(B , B')

How to assess $\Delta(A, A')$:

I Choose a costfunction on outcomes: $c(x, y)$

II Get a distance for probabilities (Wasserstein)

$$W_1(p, p') := \inf_{\gamma \in \Gamma(p,p')} \sum_{x,y \in \mathcal{A}} c(x, y)\gamma(x, y)$$

III Get a distance for POVMs

$$\Delta_M(A, A') := \sup_{\rho} W_1(p_\rho^A, p_\rho^{A'})$$

I: What is a good $c(x, y)$ for your outcomes?

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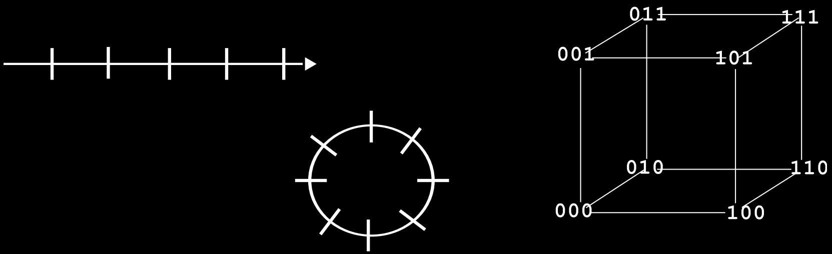

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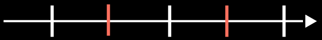

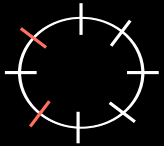

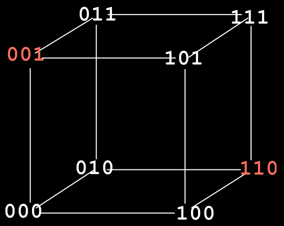

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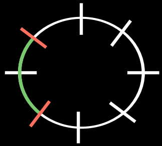

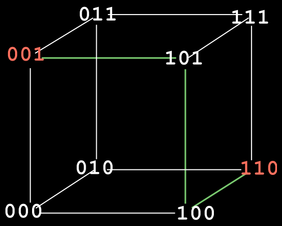

II: Wasserstein distance $W_1(p, p')$

(Transport, Kantorovic-Rubinstein, Monge, ...)

Transport Problem:

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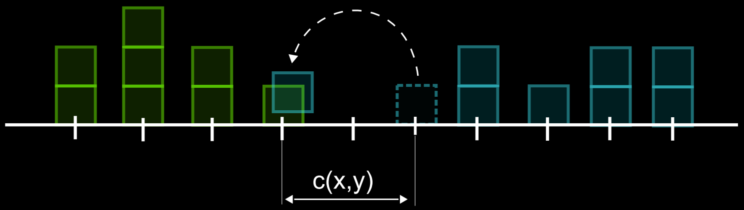
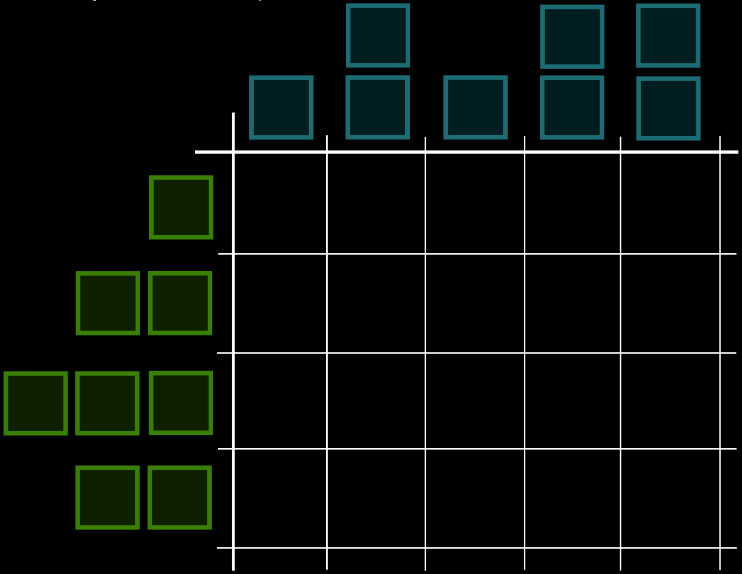
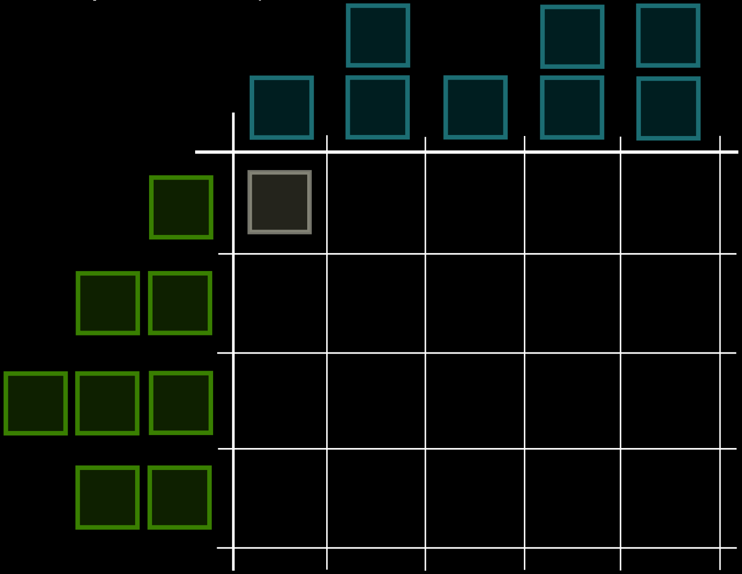


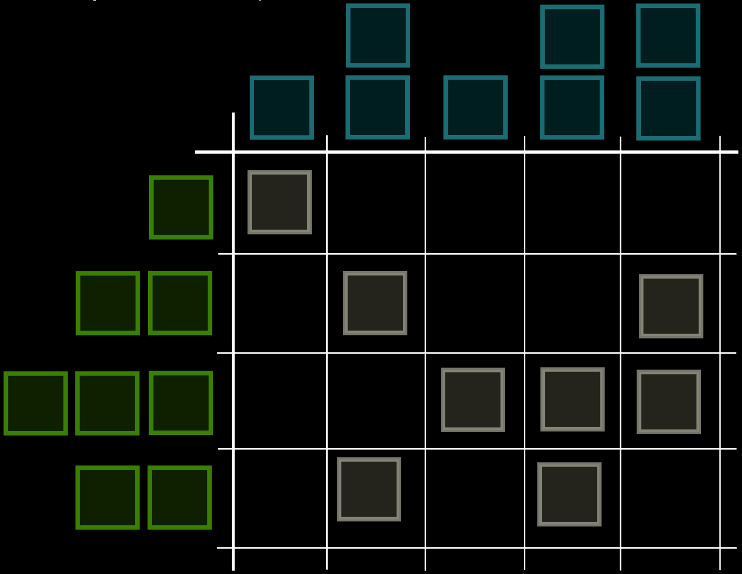
Transport Plan γ:



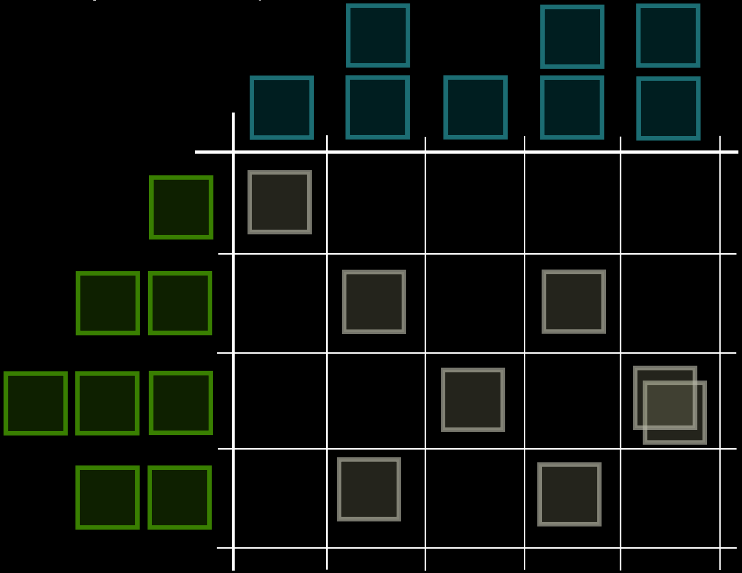
Transport Plan γ:



Transport Plan γ:

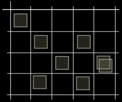


Transport Plan γ:

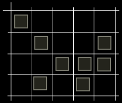


$$W_1(p, p') := \inf_{\gamma \in \Gamma} \sum_{x,y \in \mathcal{A}} \gamma(x, y) c(x, y)$$

$\Gamma(p, p') := \Bigg\{$



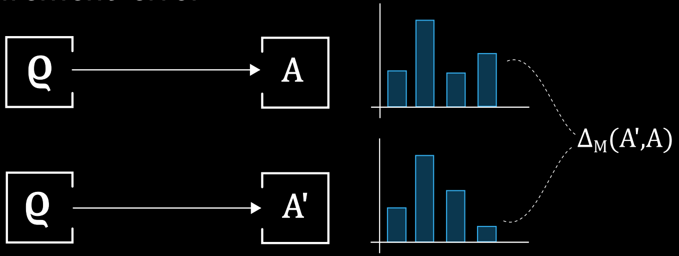
,



$, \ldots \Bigg\}$

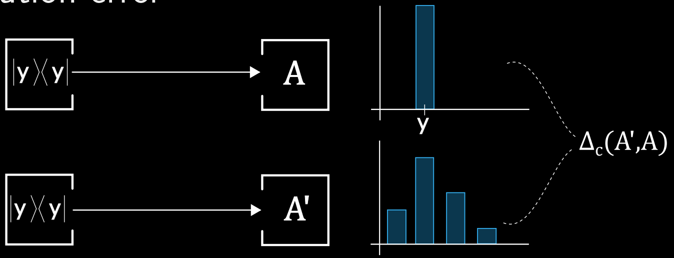
III: State independent distances $\Delta(A, A')$

Measurement error



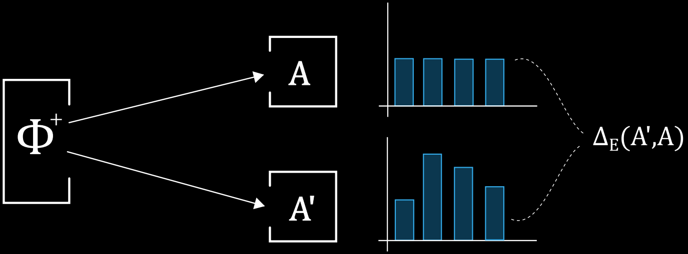
$$\Delta_M(A, A') := \sup_{\rho} W_1(p_\rho^A, p_\rho^{A'})$$

Calibration error

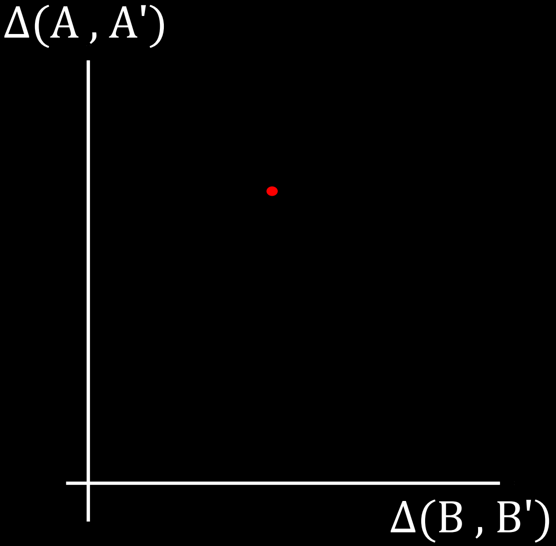


$$\Delta_C(A, A') := \sup_y W_1\left(p^{A}_{|y\rangle\langle y|}, p^{A'}_{|y\rangle\langle y|}\right)$$

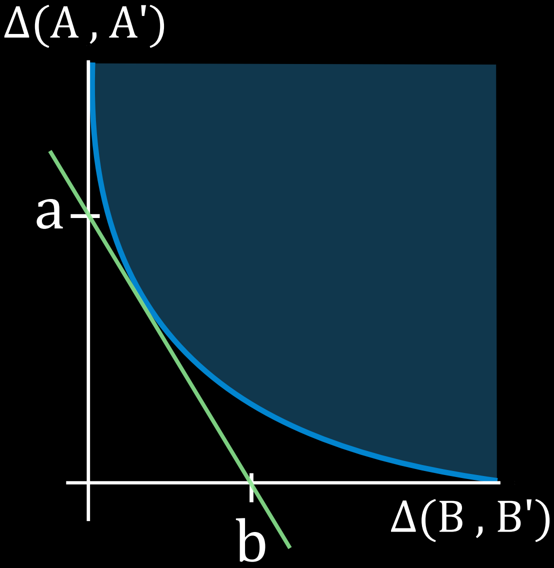
Entangled reference error



$$\Delta_C(A, A') := \frac{1}{d} \sum_{x,y} \mathrm{tr}\left[P(x) P'(y)\right] c(x, y)$$

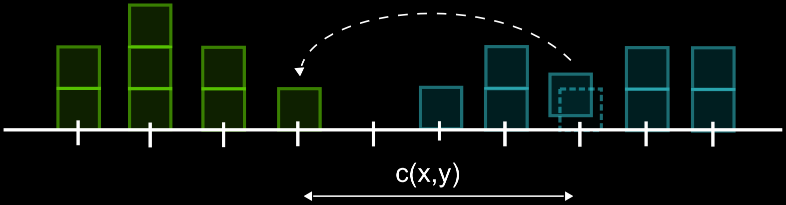


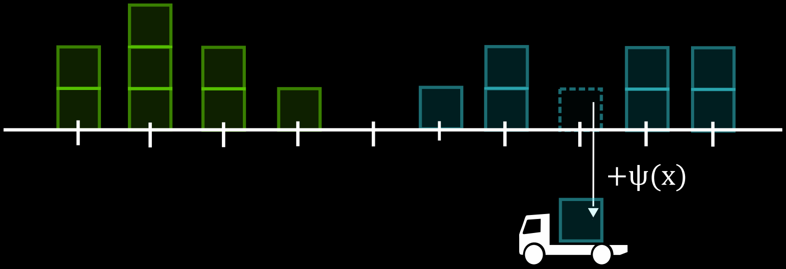
$$\mathcal{E}(a,b) = \inf_C a\Delta(A,A') + b\Delta(B,B')$$

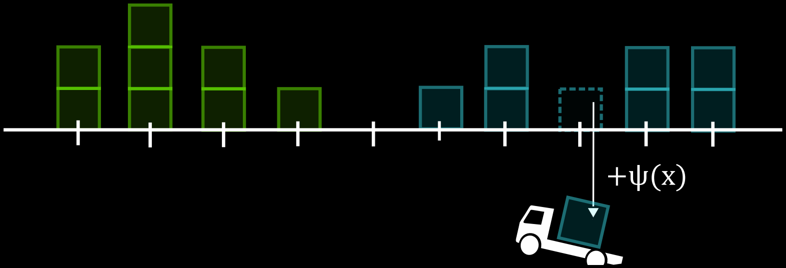


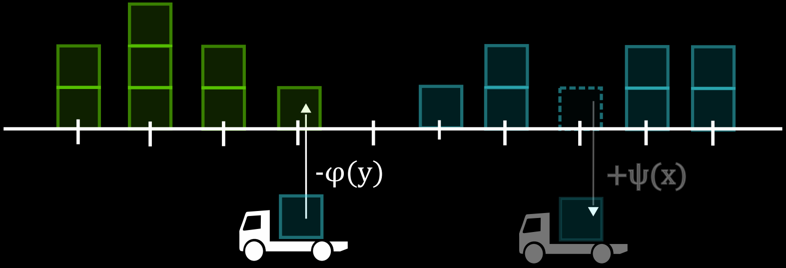
$$\mathcal{E}(a,b) = \inf_{C} a\Delta(A,A') + b\Delta(B,B')$$

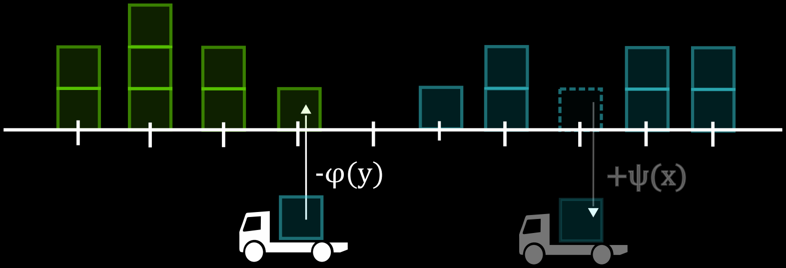
Kantorovic duality


c(x,y)


$+\psi(x)$


+ψ(x)


-φ(y)
+ψ(x)



$$\psi(x) - \varphi(y) \leqslant c(x, y)$$

"pricing"

$$\sum_{x,y} \gamma(x,y) c(x,y) \geqslant \sum_{x,y} \gamma(x,y)(\psi(x) - \varphi(y))$$
$$= \sum_{x} p(x)\psi(x) - \sum_{y} p'(y)\varphi(y)$$

Duality:

$$\inf_{\gamma \in \Gamma(p,p')} \sum_{x,y} \gamma(x,y) c(x,y)$$

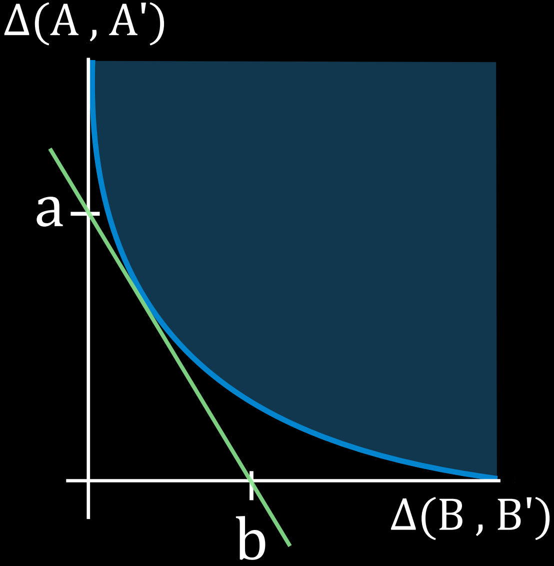
$$= \sup_{(\psi,\varphi)} \sum_{x} p(x)\psi(x) - \sum_{y} p'(y)\varphi(y)$$

Equality if $\psi(x) - \varphi(y) = c(x, y)$ on *supp*(γ)

$$[\cdots]$$

consider only

$$\{(\psi_\alpha, \varphi_\alpha)\}_{\alpha\in\Omega} \quad |\Omega| < \infty$$



$$\mathcal{E}(a,b) = \inf_{C} a\Delta_M(A,A') + b\Delta_M(B,B')$$

$$\mathcal{E}_M(a,b) = \inf_{C} \max_{\alpha,\beta} \sup_{\rho,\sigma}$$
$$a \sum_{x,y\in\mathcal{A}} \left(\psi_\alpha(x)\,\mathrm{tr}[\rho P(x)] - \varphi_\alpha(y)\,\mathrm{tr}[\rho P'(y)]\right) +$$
$$b \sum_{x,y\in\mathcal{B}} \left(\psi_\beta(x)\,\mathrm{tr}[\sigma Q(x)] - \varphi_\beta(y)\,\mathrm{tr}[\sigma Q'(y)]\right)$$

$$\mathcal{E}_M(a,b) = \inf_C \max_{\alpha,\beta} \quad a \left\| \sum_{x,y\in\mathcal{A}} (\psi_\alpha(x)P(x) - \varphi_\alpha(y)P'(y)) \right\|_\infty + b \left\| \sum_{x,y\in\mathcal{B}} (\psi_\beta(x)Q(x) - \varphi_\beta(y)Q'(y)) \right\|_\infty$$

$$\mathcal{E}_M(a,b) = \inf a\mu + b\lambda$$

subj. to

$$\mu\mathbb{I} \geqslant \sum_{x\in\mathcal{A}} \psi_\alpha(x)P(x) - \sum_{y} \phi_\alpha(y)P'(y) \quad \forall\alpha$$

$$\lambda\mathbb{I} \geqslant \sum_{x'\in\mathcal{B}} \psi_\beta(x')Q(x') - \sum_{y'\in\mathcal{B}} \phi_\beta(y')Q'(y') \quad \forall\beta$$

$$R(x,x') \geqslant 0 \qquad \forall(x,x') \in \mathcal{A}\times\mathcal{B}$$

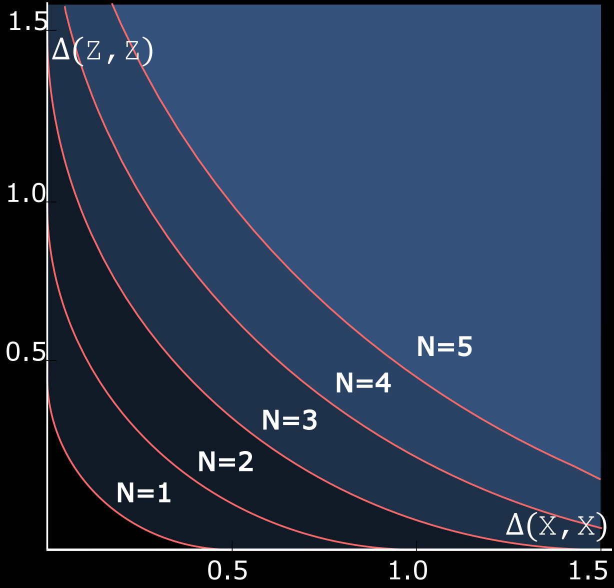
$$\sum_{x'\in\mathcal{B}} R(x,x') = P'(x) \quad \forall x \in \mathcal{A}$$

$$\sum_{x\in\mathcal{A}} R(x,x') = Q'(x') \quad \forall x' \in \mathcal{B}$$

$$\sum_{x\in\mathcal{A} x'\mathcal{B}} R(x,x') = \mathbb{I}$$

Examples

Bit-Errors



Spin-1