Qubit triple measurement uncertainty

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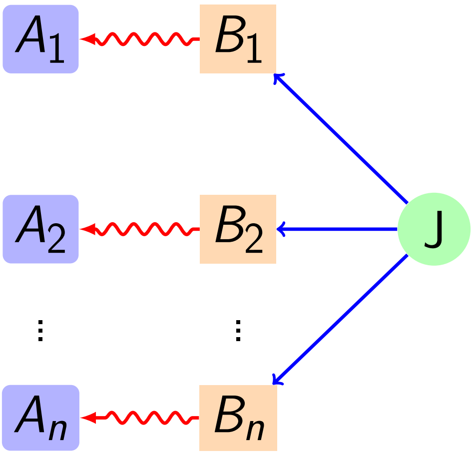
Overview

The problem

- Fix a Hilbert space $\mathcal{H}$
- Have some family $\{A_i \mid i \in 1...n\}$ of *incompatible* observables we would like to measure $A_i : \Omega_i \to \mathcal{L}(\mathcal{H})$
- Consider an arbitrary family of compatible observables, with the same outcome spaces $B_i : \Omega_i \to \mathcal{L}(\mathcal{H})$
- Ensure compatibility by requiring that the B_i are marginals of some joint $J : \prod_i \Omega_i \to \mathcal{L}(\mathcal{H})$
- Choose a figure of merit δ for an approximation and explore the set of allowed vectors $(\delta(A_1, B_1), ...\delta(A_n, B_n))$

The problem



Figures of merit

- POVM + state = probabilty distribution
- Statisticians know many ways of measuring similarity of probability distributions
- Here we take the worst case difference of the probabilities
- Symbolically

$$d(P, Q) = \sup_{\omega \in \Omega} |P(\omega) - Q(\omega)| \tag{1}$$

- Which state to use? -The worst one!
- Sup "norm" of a POVM

$$||E||_{\text{sup}} := \sup_{\rho} \sup_{\omega \in \Omega} |\text{tr}(E(\omega)\rho)| \tag{2}$$

$$d(E, F) = ||E - F||_{\text{sup}} \tag{3}$$

The difficult bit

- Exploring the space of joints is hard
- $J : \prod_i \Omega_i \to \mathcal{L}(\mathcal{H})$ is often a POVM with very many outcomes
- Explicit parameterisations are not known
- Sometimes we can impose *covariance* to reduce the search space
  - Dammeier, Schwonnek and Werner NJP 1709.3046
  - Carmeli, Heinosaari, Reitzner, Schultz and Toigo Mathematics 2016 4 54
  - Busch, Kiukas and Werner arXiv:1604.00566
  - many others

Covariance (1)

- Given a group G, with an action $(\cdot)$ on a set Ω, and an (anti-)unitary projective representation $\{U_g \mid g \in G\}$ acting on Hilbert space $\mathcal{H}$ we say an observable $E : \Omega \to \mathcal{L}(\mathcal{H})$ is *covariant* if

$$E(g \cdot \omega) = U_g E(\omega) U_g^*, \quad \forall g \in G, \omega \in \Omega \tag{4}$$

- We can't require this in general, but covariance is often present in physically relevant scenarios
- For all self-adjoint operators ρ, and for all $g, h \in G$ we have

$$U_g U_h \rho U_h^* U_g^* = U_{gh} \rho U_{gh}^* \tag{5}$$

Covariance (2)

- Given G and U_g we can define the *group averaging map*, which maps POVMs to POVMs

$$M(E)(\omega) = \frac{1}{|G|} \sum_{g \in G} U_{g^{-1}} E(g.\omega) U^*_{g^{-1}} \tag{6}$$

- Covariant observables are invariant under M
- It is easy to verify that $M(E)$ is always covariant
- Under an additional (natural) assumption M also acts to reduce the sup-norm of a POVM: $\forall \omega, \omega' \in \Omega$

$$\left| \{ g \in G \,|\, g.\omega = \omega' \} \right| = \frac{|G|}{|\Omega|} \tag{7}$$

Qubit orthogonal triple

- Attempting to simultaneously approximate the observables

$$A_\pm = \frac{1}{2}\left(1 \pm \vec{a}\cdot\vec{\sigma}\right) \tag{8}$$

$$B_\pm = \frac{1}{2}\left(1 \pm \vec{b}\cdot\vec{\sigma}\right) \tag{9}$$

$$C_\pm = \frac{1}{2}\left(1 \pm \vec{c}\cdot\vec{\sigma}\right), \tag{10}$$

  where $\vec{a}$, $\vec{b}$, $\vec{c}$ are pairwise orthogonal
- Column vectors will be written in the $\vec{a}$, $\vec{b}$, $\vec{c}$ basis so

$$\vec{a} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \vec{b} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \vec{c} = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{11}$$

Approximators

- Define

$$D_k = \frac{1}{2}\left(1 + kd_0 + k\vec{d}\cdot\vec{\sigma}\right) = \sum_{l,m} J_{klm} \tag{12}$$

$$E_l = \frac{1}{2}\left(1 + ld_0 + l\vec{d}\cdot\vec{\sigma}\right) = \sum_{k,m} J_{klm} \tag{13}$$

$$F_m = \frac{1}{2}\left(1 + md_0 + m\vec{d}\cdot\vec{\sigma}\right) = \sum_{k,l} J_{klm}, \tag{14}$$

  where $k, l, m \in \{+1, -1\}$
- We must impose the constraints $J_{klm} \geq 0$ and $\sum_{klm} J_{klm} = 1$

What group should we use?

- We need $\frac{|G|}{|\Omega|} \in \mathbb{Z}$, so look for an 8 element group
- A natural choice is given by the elementary Abelian group E8 $\cong (\mathbb{Z}/2\mathbb{Z})^3$
- We can label each group element with a tuple of three numbers, each either 1 or -1 then

$$g(h,i,j)\, g(k,l,m) = g(hk, il, jm) \tag{15}$$

- The group action on Ω is similar

$$g(h,i,j) \cdot (k,l,m) = (hk, il, jm) \tag{16}$$

Representation

- It is easy to verify that the following assignments give a projective representation of E8 with the required properties

$$U_{g(+,+,+)} = \mathsf{I} \qquad U_{g(-,-,-)} = \Gamma \tag{17}$$

$$U_{g(+,-,-)} = \mathsf{X} \qquad U_{g(-,+,+)} = \Gamma\,\mathsf{X} \tag{18}$$

$$U_{g(-,+,-)} = \mathsf{Y} \qquad U_{g(+,-,+)} = \Gamma\,\mathsf{Y} \tag{19}$$

$$U_{g(-,-,+)} = \mathsf{Z} \qquad U_{g(+,+,-)} = \Gamma\,\mathsf{Z} \tag{20}$$

- Γ is an *anti-unitary* operator obeying $\Gamma\,(\mathsf{I} + \vec{r}\cdot\vec{\sigma})\,\Gamma^* = \mathsf{I} - \vec{r}\cdot\sigma$, $\forall r \in \mathbb{R}^3$

The main result (1)

- We consider a different group action depending on which marginal we are looking at
- For example, for the first marginal we use $g(k,l,m) \cdot h = kh$, for the second $g(k,l,m) \cdot i = li$, etc.
- These marginal actions obey all the assumptions we need, and the target measurements are covariant so

$$M(A) = A \tag{21}$$
$$M(B) = B \tag{22}$$
$$M(C) = C \tag{23}$$
$$\tag{24}$$

The main result (2)

- In particular

$$d(M(D), A) = d(M(D), M(A)) \tag{25}$$
$$= ||M(D - A)||_{\text{sup}} \tag{26}$$
$$\leq ||D - A||_{\text{sup}} \tag{27}$$
$$= d(D, A), \tag{28}$$

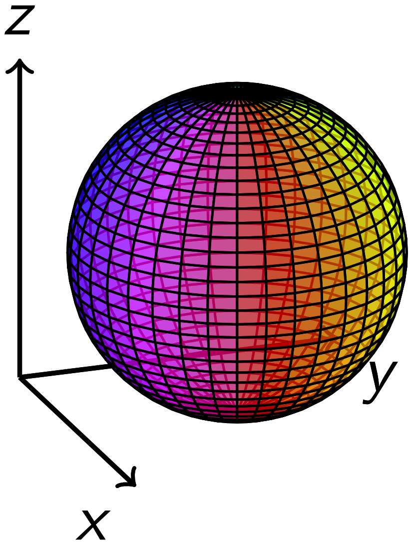
- and similar for the B, E, and C, F pairs
- Applying the map to a joint observable therefore does not increase the error of any of the marginals

The main result (3)

- Covariance fixes the form of J

$$J_{k,l,m} = \frac{1}{8}\left(\mathbb{I} + \begin{pmatrix} k\,j_x \\ l\,j_y \\ m\,j_z \end{pmatrix} \cdot \vec{\sigma}\right) \tag{29}$$

- where j_x, j_y and j_z may be chosen freely as long as $j_x^2 + j_y^2 + j_z^2 \leq 1$
- Computing the marginals then gives $d(D, A) = \frac{1}{2}(1 - j_x)$, and similar for y and z
- The set of allowed $(d(D, A), d(E, B), d(F, C))$ values is therefore a sphere of radius $\frac{1}{2}$, centered at point $\frac{1}{2}$


z
y
x

Thank you for your time
and hopefully your attention!

Covariance of group averaging mapped observable

$$M(E)(\omega) = \frac{1}{|G|} \sum_{g \in G} U_{g^{-1}} E(g.\omega) U^*_{g^{-1}} \tag{30}$$

Let $\tilde{g}h = g$

$$M(E)(\omega) = \frac{1}{|G|} \sum_{\tilde{g} \in G} U_{h^{-1}\tilde{g}^{-1}} E(\tilde{g}h.\omega) U^*_{h^{-1}\tilde{g}^{-1}} \tag{31}$$

$$= U_{h^{-1}} \left(\frac{1}{|G|} \sum_{\tilde{g} \in G} U_{\tilde{g}^{-1}} E(\tilde{g}h.\omega) U^*_{\tilde{g}^{-1}} \right) U^*_{h^{-1}} \tag{32}$$

$$\implies U_h M(E)(\omega) U^*_h = M(E)(h.\omega) \tag{33}$$

M acts to reduce the sup-norm

$$
\begin{aligned}
||M(E)||_{\text{sup}} &= \sup_{\omega} ||M(E)(\omega)|| && (34)\\
&= \sup_{\omega} \left|\left| \frac{1}{|G|} \sum_{g \in G} U_{g^{-1}} E(g.\omega) U^{*}_{g^{-1}} \right|\right| && (35)\\
&\leq \frac{1}{|G|} \sup_{\omega} \sum_{g \in G} \left|\left| U_{g^{-1}} E(g.\omega) U^{*}_{g^{-1}} \right|\right| && (36)\\
&= \frac{1}{|\Omega|} \sup_{\omega} \sum_{\omega' \in \Omega} ||E(\omega')|| && (37)\\
&\leq \sup_{\omega} ||E(\omega)|| && (38)\\
&= ||E||_{\text{sup}} && (39)
\end{aligned}
$$