Genuine *n*-wise Measurement Incompatibility and Device Independent Certificates of Incompatibility

Marco Túlio Quintino

August 31, 2017

In collaboration with:
Daniel Cavalcanti (ICFO), Costantino Budroni (Vienna), Adán Cabello (Sevilla)
and
Flavien Hirsch (GAP), Nicolas Brunner (GAP), Joseph Bowles (ICFO)
PRA (2016) + arXiv (2017)

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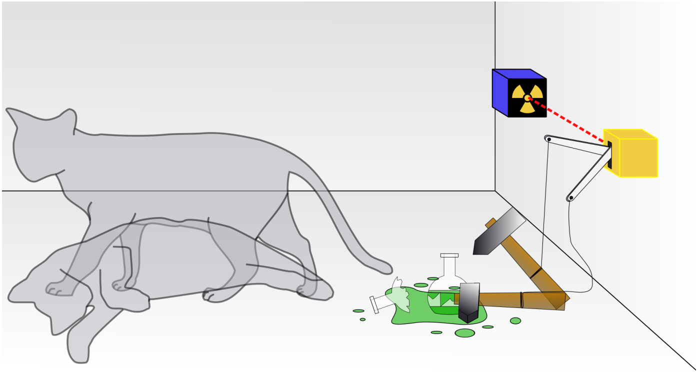
Quantum Mechanics



States and Measurements

$$\rho_{AB} \neq \int \pi(\lambda) \rho_A^\lambda \otimes \rho_B^\lambda \, \mathrm{d}\lambda$$

$$\Delta x \, \Delta p \geq \hbar/2$$

Erwin Schrödinger

Werner Heisenberg

Born's Rule

$$p(ab|xy) = \mathrm{tr}(\rho_{AB} A_{a|x} \otimes B_{b|y})$$

Max Born

Measurement Incompatibility

$$\Delta x \, \Delta p \geq \hbar/2$$

Compatible Measurements

- Quantum observables:

$$E = E^\dagger, \qquad F = F^\dagger$$

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- Joint Measurability

More general measurements

- POVM:

$$E_e \geq 0, \quad \sum_e E_e = I$$

$$F_f \geq 0, \quad \sum_f F_f = I$$

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- Commutation of the POVM elements?

Joint Measurability

- $\{E_e\}$ and $\{F_f\}$ are JM if there exists a third measurement $\{G_{ef}\}$, such that

$$E_e = \sum_f G_{ef}, \quad F_f = \sum_e G_{ef}$$

Joint Measurability

- $\{E_e\}$ and $\{F_f\}$ are JM if there exists a third measurement $\{G_{ef}\}$, such that

$$E_e = \sum_f G_{ef}, \quad F_f = \sum_e G_{ef}$$

- By measuring $\{G_{ef}\}$ we get the output e and f

Pauli Measurements

$$\sigma_Z : \{|0\rangle\langle 0|\,,\ |1\rangle\langle 1|\} \quad \sigma_X : \{|+\rangle\langle +|\,,\ |-\rangle\langle -|\}$$

Noise Pauli Measurements

$$\sigma_{Z,\eta} : \left\{ \eta \left|0\right\rangle\left\langle 0\right| + (1-\eta)\frac{I}{2} \; ; \qquad \eta \left|1\right\rangle\left\langle 1\right| + (1-\eta)\frac{I}{2} \right\}$$

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$$\eta \leq \frac{1}{\sqrt{2}} \iff \text{Joint Measurability}$$

P. Busch. Phys. Rev. D (1986)

Hollow Triangle

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$$\sigma_{Y,\eta} : \left\{ \eta \left|Y+\right\rangle\left\langle Y+\right| + (1-\eta)\frac{I}{2} \ ; \qquad \eta \left|Y-\right\rangle\left\langle Y-\right| + (1-\eta)\frac{I}{2} \right\}$$

$$\eta \leq \frac{1}{\sqrt{2}} \iff \text{Pairwise Measurability}$$

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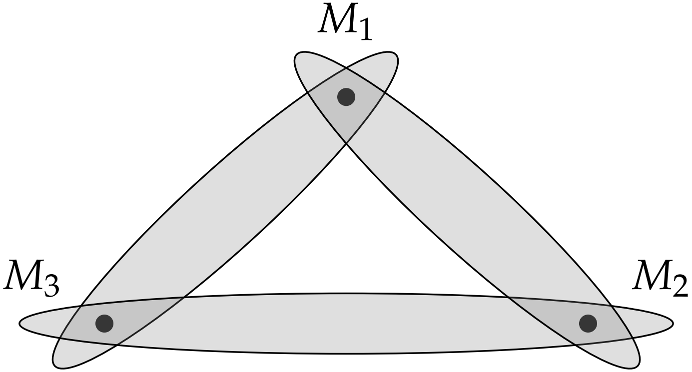
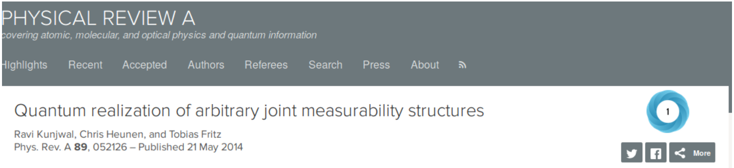
T. Heinosaari, D. Reitzner, P. Stano: Foundations of Physics (2008)

Hollow Triangle Measurements



T. Heinosaari, D. Reitzner, P. Stano: Foundations of Physics (2008)

General Measurement Compatibility


PHYSICAL REVIEW A

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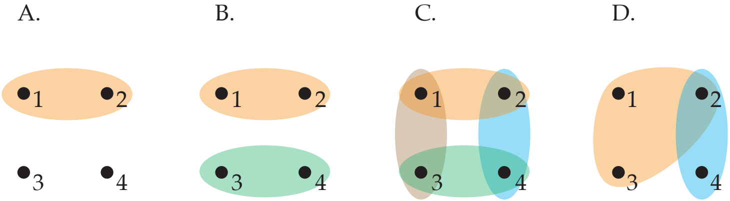
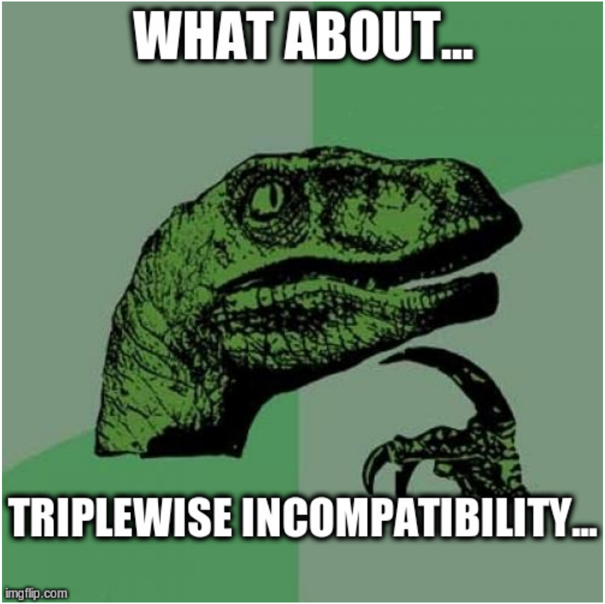
Quantum realization of arbitrary joint measurability structures

Ravi Kunjwal, Chris Heunen, and Tobias Fritz

Phys. Rev. A **89**, 052126 – Published 21 May 2014



First Question



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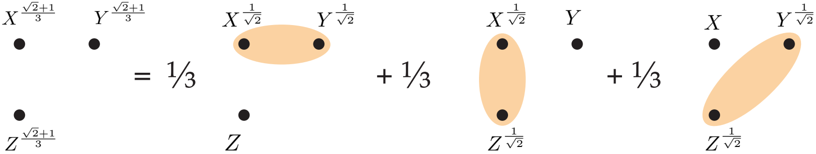
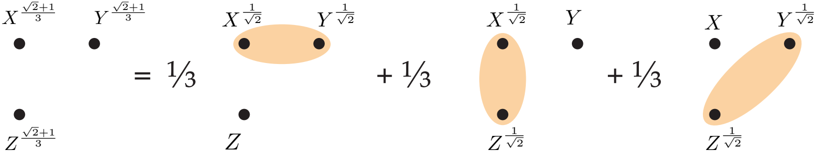
$$\eta \leq \frac{1}{\sqrt{3}} \iff \text{Triplewise Measurability}$$

Example



$X^{\frac{\sqrt{2}+1}{3}}$
$Y^{\frac{\sqrt{2}+1}{3}}$
$Z^{\frac{\sqrt{2}+1}{3}}$
= ⅓
$X^{\frac{1}{\sqrt{2}}}$
$Y^{\frac{1}{\sqrt{2}}}$
Z
+ ⅓
$X^{\frac{1}{\sqrt{2}}}$
Y
$Z^{\frac{1}{\sqrt{2}}}$
+ ⅓
X
$Y^{\frac{1}{\sqrt{2}}}$
$Z^{\frac{1}{\sqrt{2}}}$

Example



Non-genuine triplewise compatible measurements:

$$A_{a|x} = p_{12} J^{12}_{a|x} + p_{23} J^{23}_{a|x} + p_{13} J^{13}_{a|x}$$

Hollow Triangle

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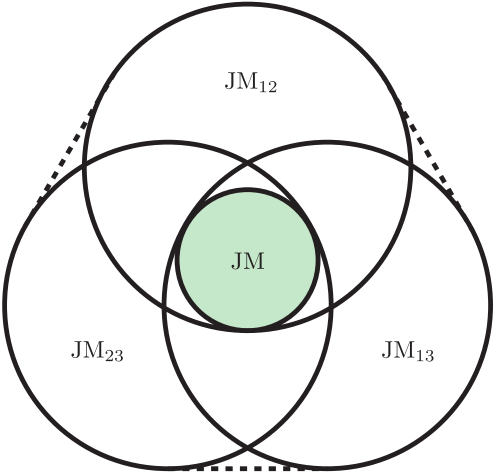
$$\eta \leq \frac{1}{\sqrt{2}} \approx 0.707 \iff \text{Pairwise Measurability}$$

$$\eta \leq \frac{1}{\sqrt{3}} \approx 0.577 \iff \text{Triplewise Measurability}$$

$$\eta > \frac{\sqrt{2}+1}{3} \approx 0.805 \iff \text{Genuine Triplewise incompatibility}$$

Geometrical Interpretation



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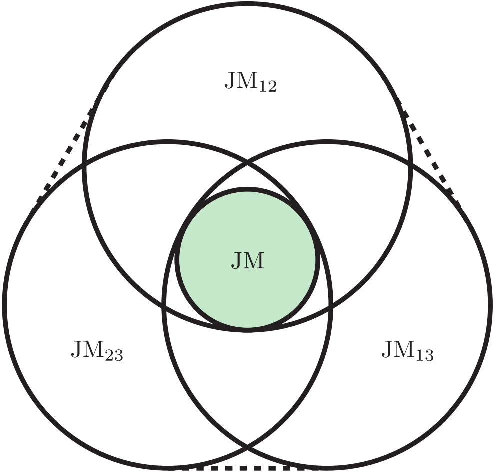
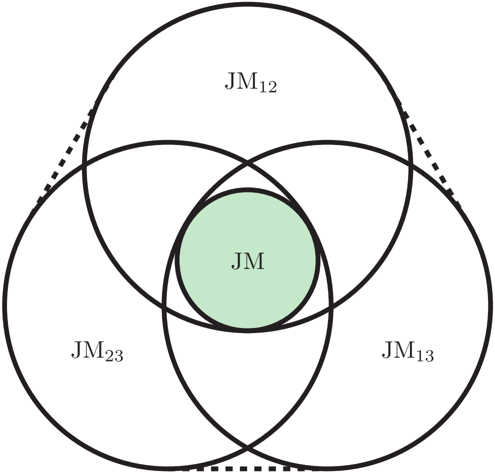


$$A_{a|x} = p_{12} J^{12}_{a|x} + p_{23} J^{23}_{a|x} + p_{13} J^{13}_{a|x}$$

Geometrical Interpretation



(All these sets admits an SDP characterisation)

Incompatibility Witness

$$M_i := M_{0|i} - M_{1_i}$$

$$\mathrm{tr}(\sigma_X M_1 + \sigma_Y M_2 + \sigma_Z M_3)$$

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$$\mathrm{tr}(\sigma_X M_1 + \sigma_Y M_2 + \sigma_Z M_3) \overset{3JM}{\leq} 2(\sqrt{2} + 1) \approx 4.82$$

Genuine N-wise incompatibility

Genuine N-wise incompatibility

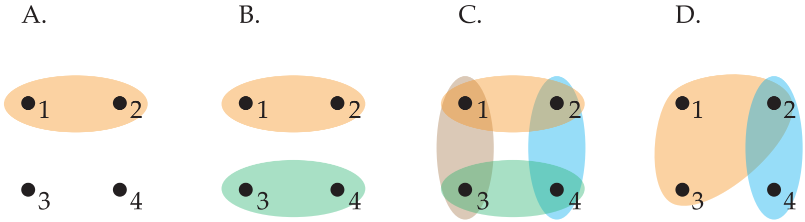
Genuine N-wise incompatibility

Genuine N-wise incompatibility . . .

Genuine N-wise incompatibility

Genuine N-wise incompatibility . . .

and more!



General Definition

Definition

Given a set of compatibility $\mathcal{C} = \{C_1, C_2, \ldots, C_N\}$, a set measurements $\{A_{a|x}\}$ is genuine $\mathcal{C}$-incompatible when it cannot be written as convex combinations of measurements that respect the compatibility C_1, C_2, . . . , and C_N.

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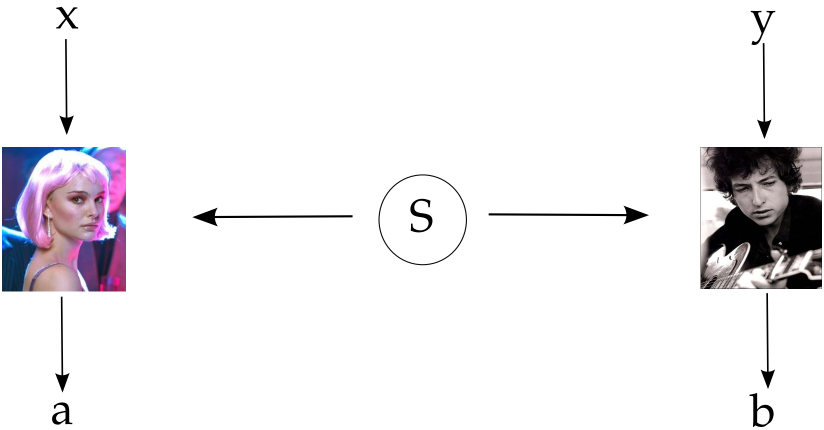
More specifically, let $\left\{J_{a|x}^{C_i}\right\}$, be a set of of measurements respecting the compatibility structure C_i. The set $\{A_{a|x}\}$ is not genuine $\mathcal{C}$-incompatible if it can be written as

$$A_{a|x} = \sum_i p_i J_{a|x}^{C_i} \tag{1}$$

for some probabilities p_i.

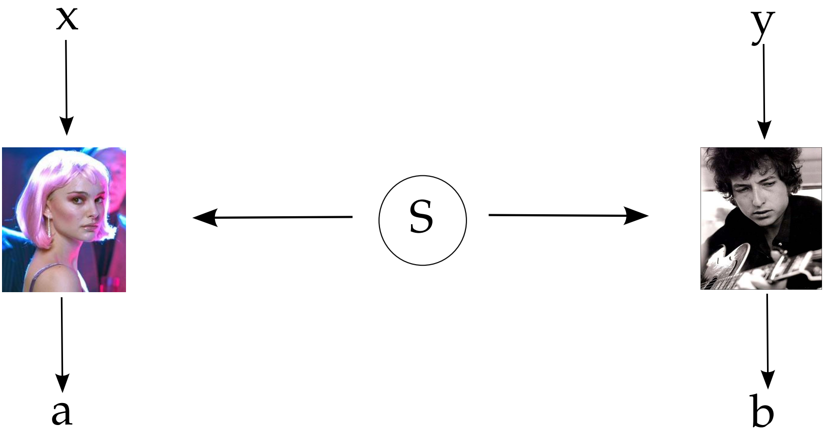

1P
35001
50000
2P
1
KO
RYU
99
KEN
ROUND 2
BATTLE 01

Bell Nonlocality



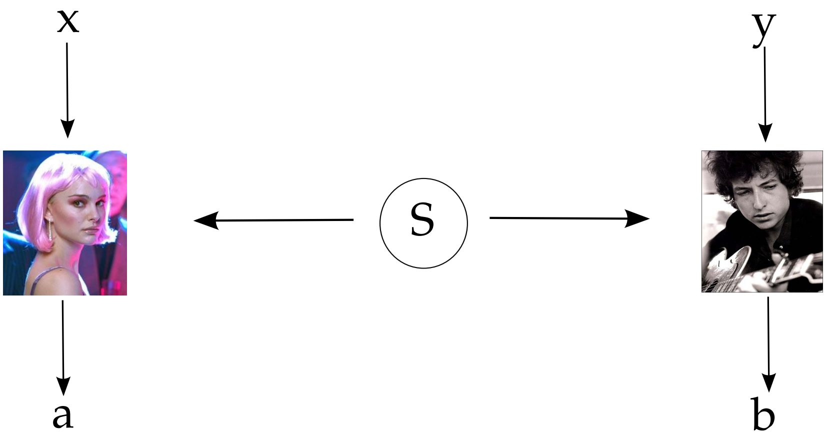
$p(ab|xy)$

Bell Nonlocality



$$p(ab|xy) = \mathrm{tr}(\rho_{AB} A_{a|x} \otimes B_{b|y})$$

Bell Nonlocality



$$p(ab|xy) = \mathrm{tr}(\rho_{AB} A_{a|x} \otimes B_{b|y})$$

$$p(ab|xy) = \sum_\lambda \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$

Bell Nonlocality

Compatible measurements $\implies$ Bell Locality

Bell Nonlocality

$$\text{Measurement Compatibility} \implies \text{Bell Locality}$$

$$\text{Bell Nonlocality} \implies \text{Measurement Incompatibility}$$

Bell Nonlocality

Measurement Compatibility $\implies$ Bell Locality

Bell Nonlocality $\implies$ Measurement Incompatibility

Device independent certification of Measurement Incompatibility!

CHSH

$$CHSH = \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \overset{LHV}{\leq} 2$$

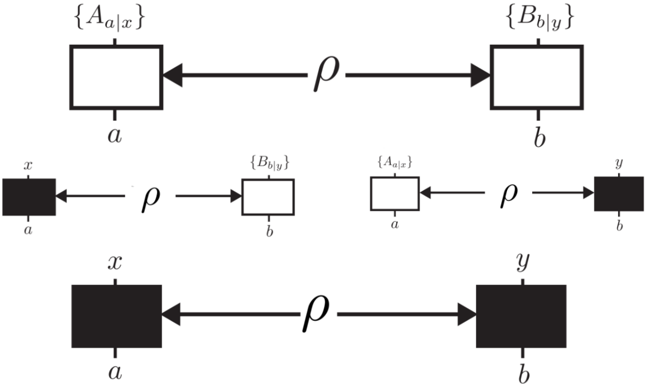
EPR Steering

σ_Z

$\longleftarrow \quad \psi^- \quad \longrightarrow$

σ_X

EPR Steering



Device Independent Certification

Can the you “certificate” the incompatibility of all measurements?

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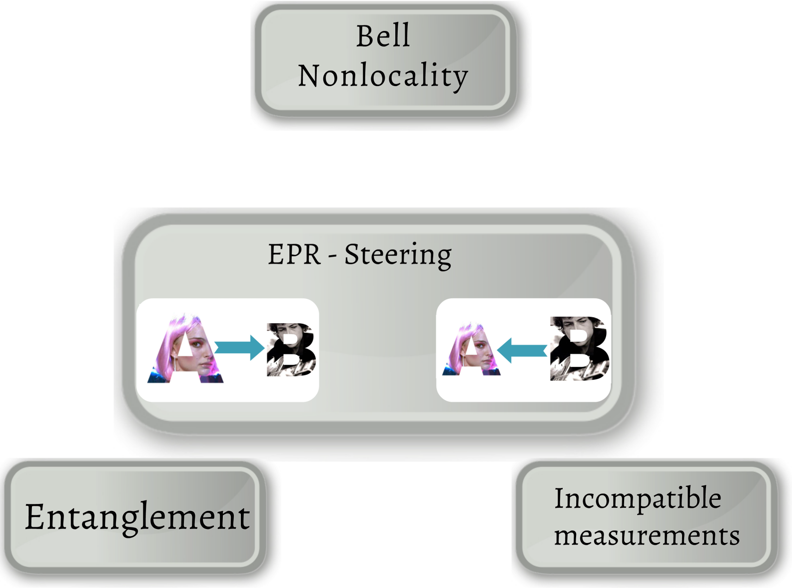
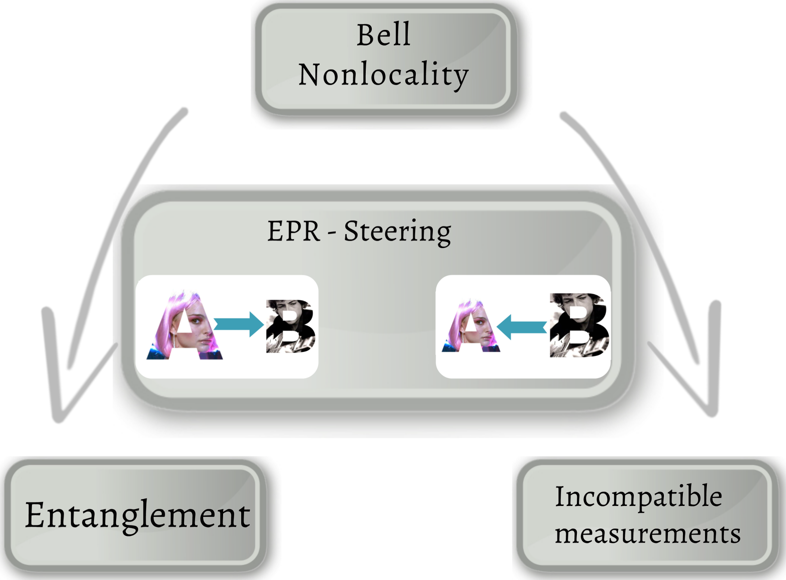
Which measurements are “useful” for Bell/EPR nonlocality?

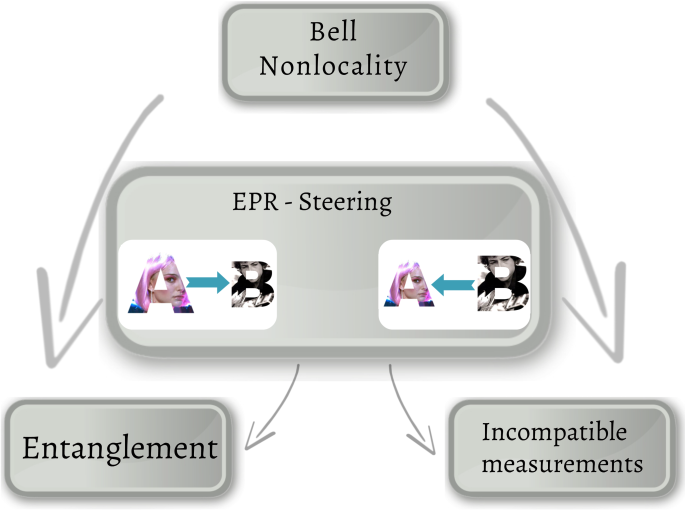
Diagram of concepts



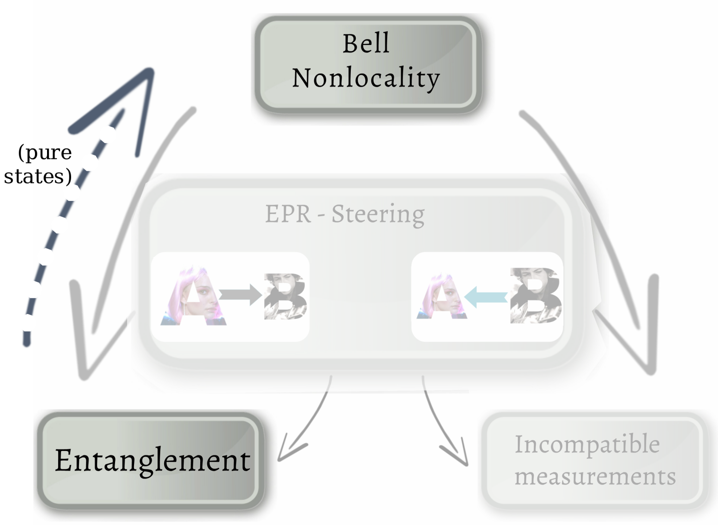
Bell Locality Requires Entanglement and Incompatible Measurements



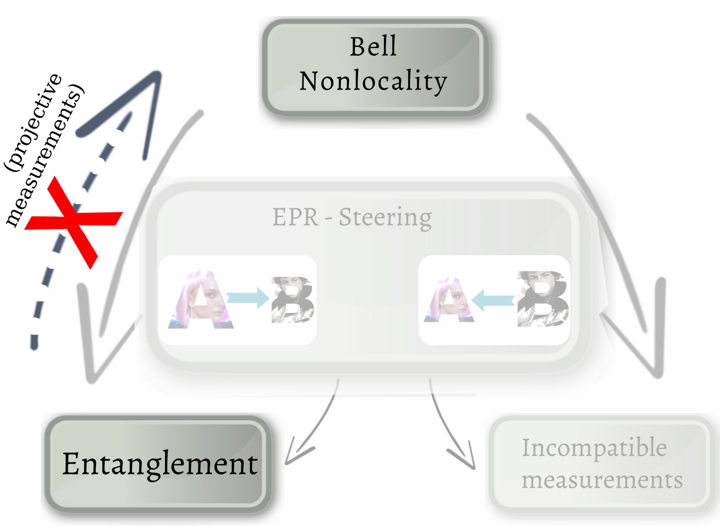
EPR-Steering Requires Entanglement and Incompatible Measurements



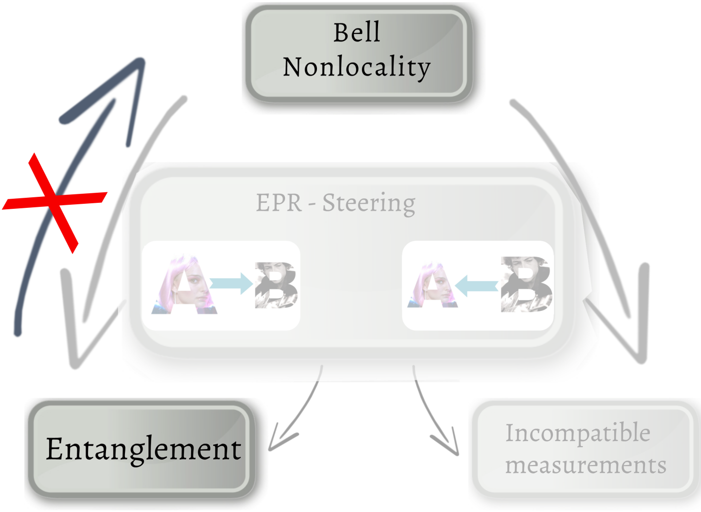
Pure states: N. Gisin (1991)



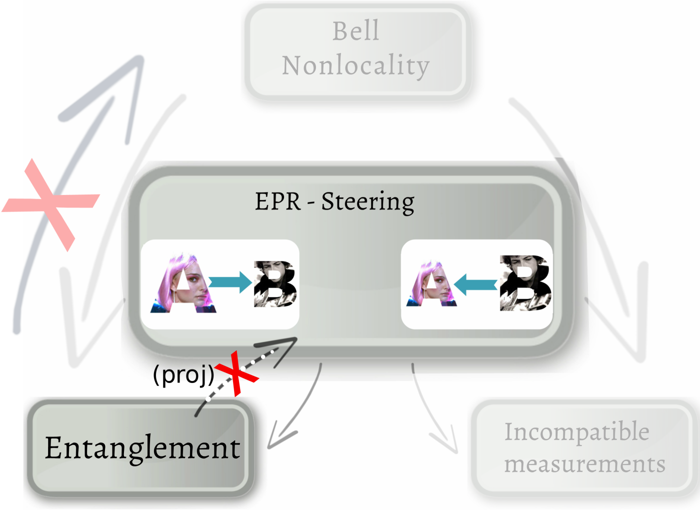
Werner States (1989)



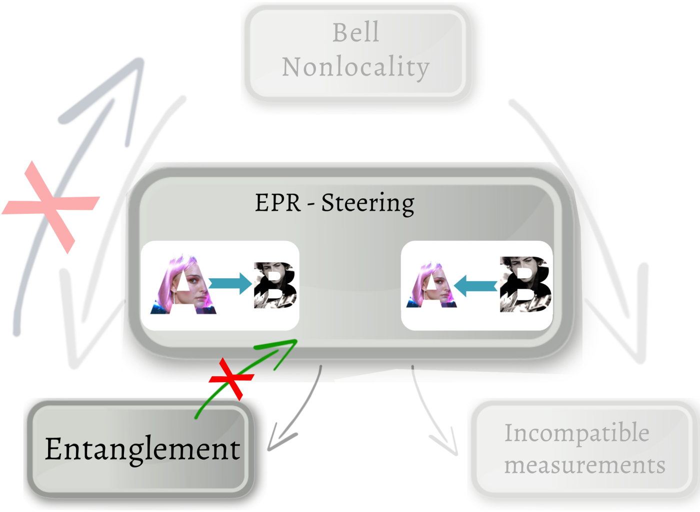
Barrett's Model (2003)



Wiseman *et al* (2007)



Quintino *et al* (2015)



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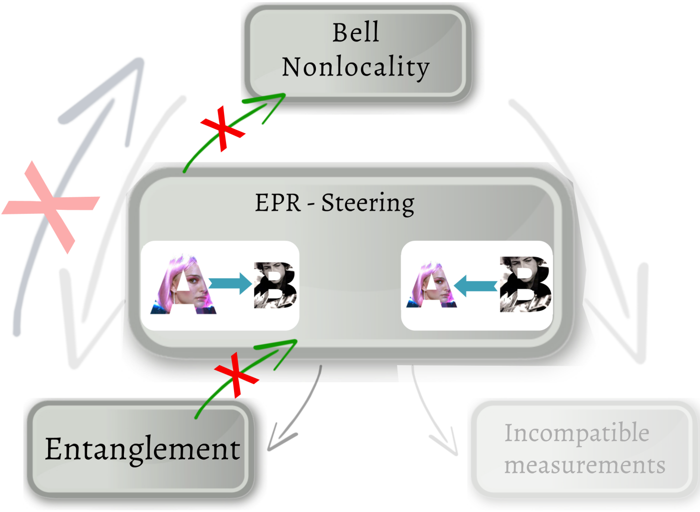
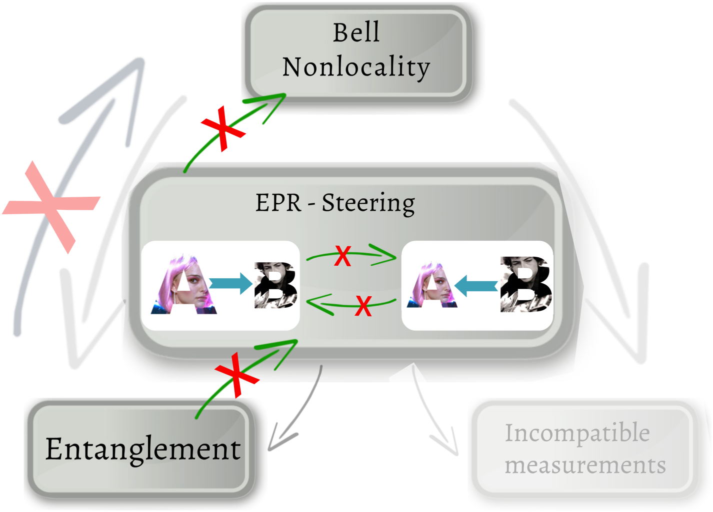
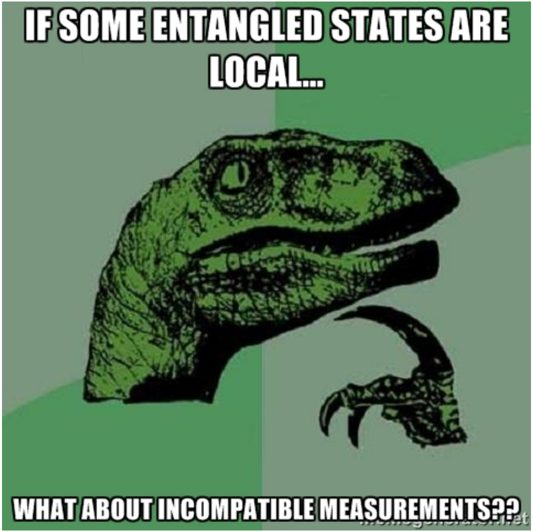


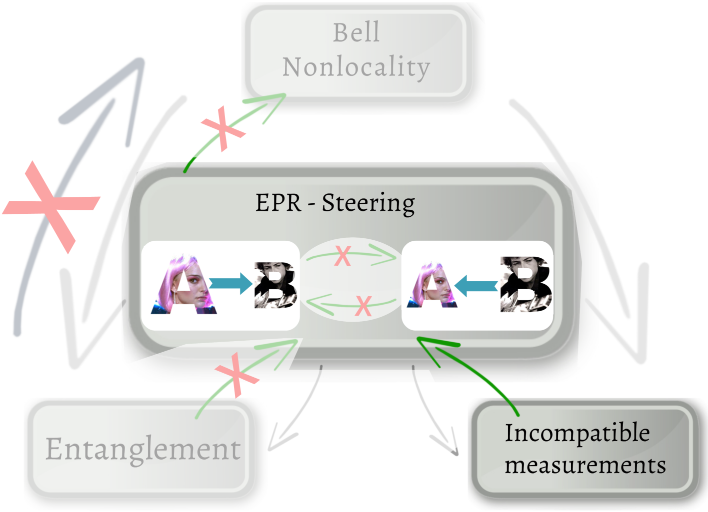
Quintino *et al* (2015)+ Bowles, Quintino, *et al* 2014



Local Incompatible Measurements??

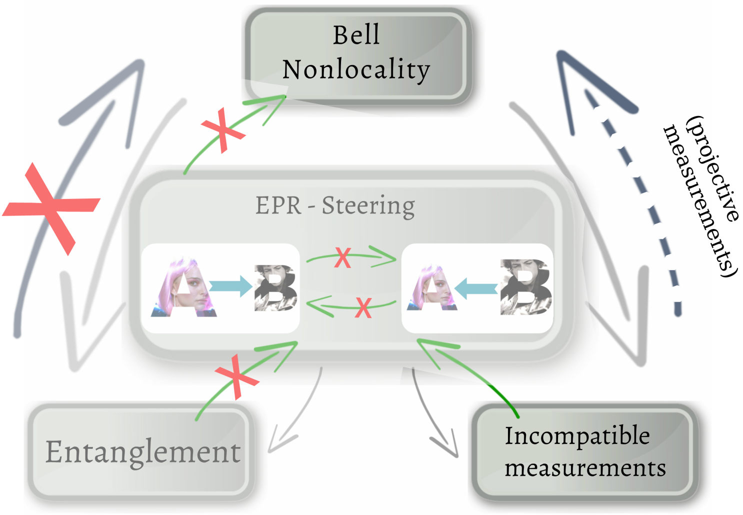


Quintino *et al* / Uola *et al* (2014)



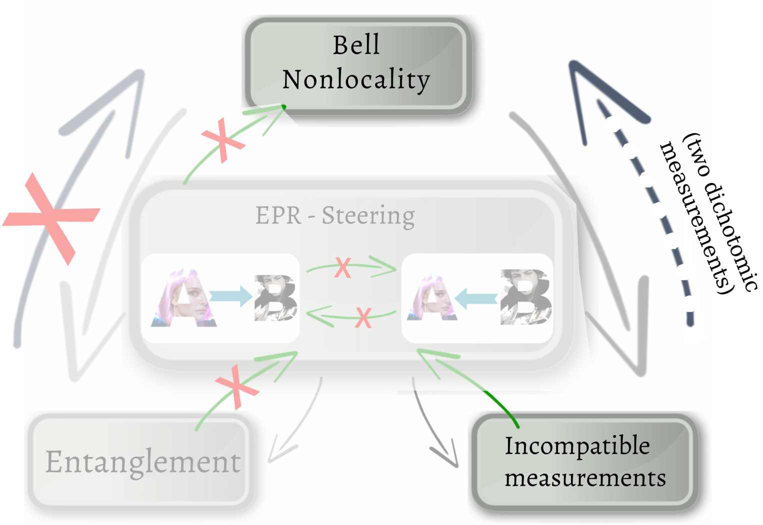
Projective Measurements

L.A. Khalfin, B.S. Tsirelson (1985)

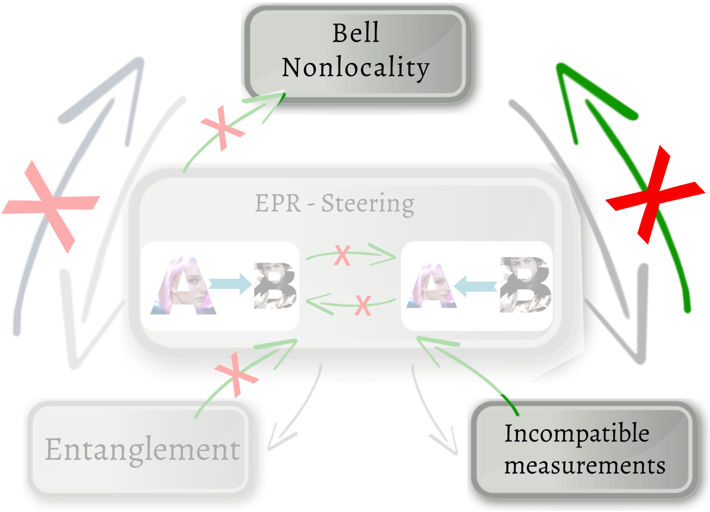


Two dichotomic measurements

M. M. Wolf, D. Perez-Garcia, C. Fernandez (2009)



Our contribution



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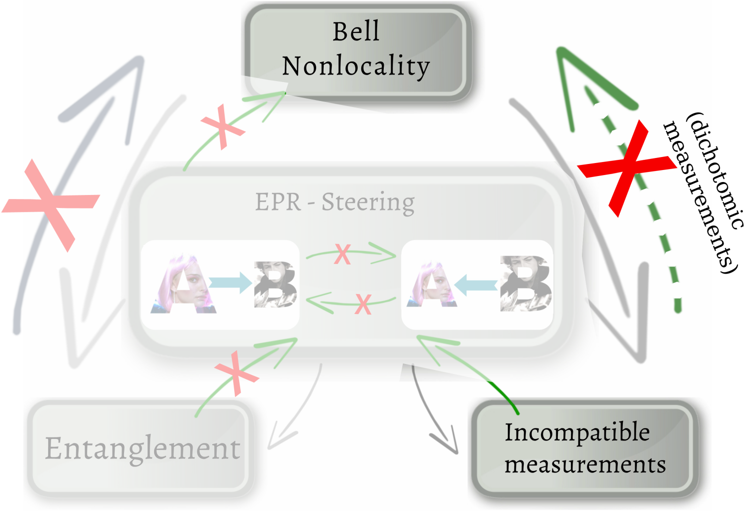

Incompatible measurements and Bell Nonlocality

Main Result

There exists a set of non Jointly Measurable measurements that can never lead to Bell nonlocality when the other part is restricted to dichotomic measurements.

PHYSICAL REVIEW A **93**, 052115 (2016)

Incompatible quantum measurements admitting a local-hidden-variable model

Marco Túlio Quintino, Joseph Bowles, Flavien Hirsch, and Nicolas Brunner

Methods

- Consider the set of all η white noise protective measurements

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Methods

- Consider the set of all η white noise protective measurements
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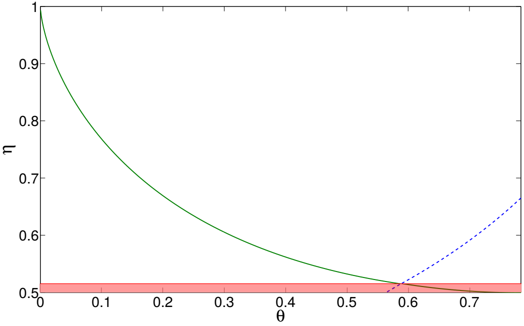
- 

The general case

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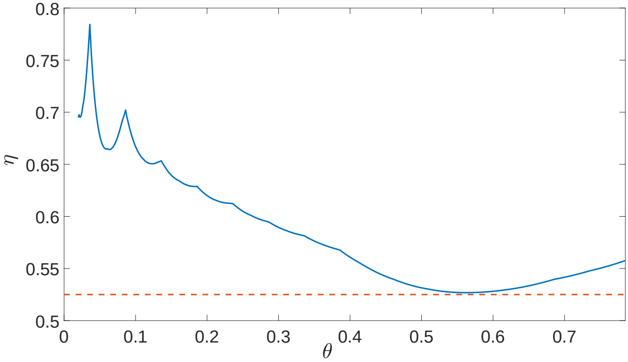
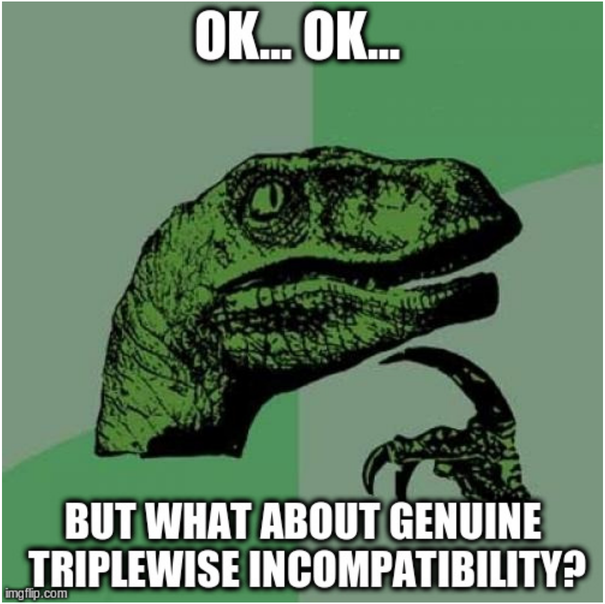
- 

Independent (but very related) work

A set of incompatible but Bell local measuremets was also presented at:
Measurement incompatibility does not give rise to Bell violation in general
Bene Erika, Tamás Vértesi
(arXiv:1705.10069)
(Similar proof techniques were used)

Device Independent Certification



Device Independent Certification

$p(ab|xy)$is Non-signalling when

$$\sum_b p(ab|xy) = \sum_b p(ab|xy')\forall a, x, y, y'$$

$$\sum_a p(ab|xy) = \sum_b ap(ab|x'y)\forall b, x, x', y'$$

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$p(ab|xy) \in L_{12}^{NS}$ is Non-signalling AND

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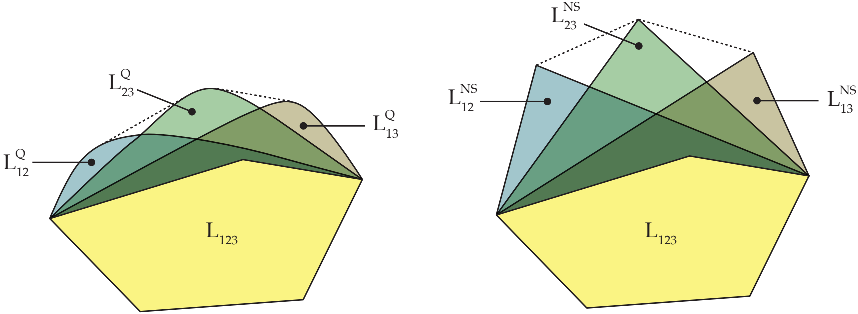
Geometry



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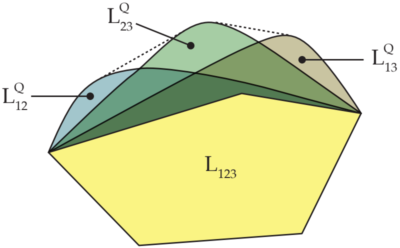
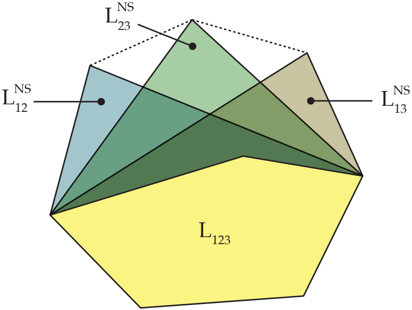


NPA hierarchy (SDP)



Linear Programming

Known Bell Inequalities

•	L	NS	NPA2	QUBIT	2L	3L
I_{3322}	0	1	0.251	0.25	0.5	0.75

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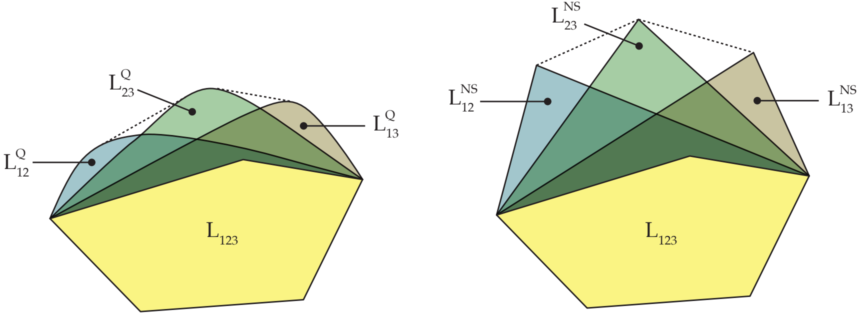
With $I_{3422}(2)$ and I_{3522} we can certify pairwise incompatibility in all pairs, but not genuine triplewise incompatibility.

Genuine 3-input NL



Full Facet Enumeration of $3L$ is possible!

Genuine 3-input NL on both sides

$$
\begin{aligned}
&-p(10|00) - p(00|01) - p(00|10) - p(00|11) \\
&-p(10|12) - p(01|20) - p(01|21) + p(00|22) \overset{3L}{\leq} 0
\end{aligned}
$$

Three Input Nonlocality

$$-p(10|00) - p(00|01) - p(00|10) - p(00|11)$$
$$-p(10|12) - p(01|20) - p(01|21) + p(00|22) \overset{3L}{\leq} 0$$

With Qutrits, one can obtain $0.34 > 0$

Semi-device independent?

Genuine 3-input steering!

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- Non-trivial Bell-nonlocality breaking channels!

Future

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Future

- Information protocols exploiting genuine n-wise incompatibility/nonlocality/etc
- Genuine triplewise incompatible but not genuine triplewise Bell-Nonlocal
- In quantum mechanics, we have genuine n-wise incompatible measurements $\forall n \in \mathbb{N}$
- Obtain a "proper" computer assisted proof for local incompatible measurements

Thank you!