Workshop on Quantum Incompatibility

Quantum steering with positive operator valued measures

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Maria Laach, August 2017

The Einstein–Podolsky–Rosen (EPR) experiment

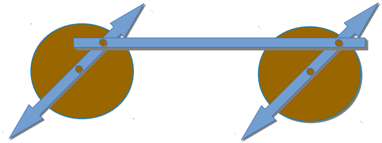
Consider the Bell state

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}(|z,+\rangle|z,-\rangle - |z,-\rangle|z,+\rangle) = \frac{1}{\sqrt{2}}(|x,+\rangle|x,-\rangle - |x,-\rangle|x,+\rangle)$$

- ☞ if Alice measures σ_z, B is 'collapsed' to $|z,+\rangle$ or $|z,-\rangle$
- ☞ if Alice measures σ_x, B is 'collapsed' to $|x,+\rangle$ or $|x,-\rangle$

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}(|\hat{n},+\rangle|\hat{n},-\rangle - |\hat{n},-\rangle|\hat{n},+\rangle)$$

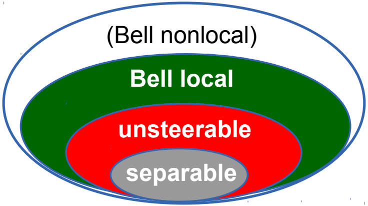
- ☞ if Alice measures $\hat{n}$, B is 'collapsed' to $|\hat{n},+\rangle$ or $|\hat{n},-\rangle$



Alice can 'steer' Bob's system into different ensembles from a distance!

Different notions of quantum nonlocality

- ☞ **EPR, 1935:** 'spooky at a distance'!?
- ☞ **Schrödinger, 1935:** 'entangled' systems express 'steering'!



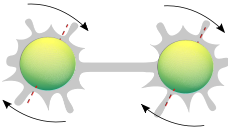
☞ **Bell nonlocality:** *certain quantum correlation is stronger than any classical correlation (Bell, 1964)*
☞ **Nonseparability:** *certain quantum states cannot be prepared by Local Operations and Classical Communication (Werner, 1989)*
☞ **Steerability:** *certain EPR experiments cannot be locally simulated (Wiseman et al., 2007)*

The verification protocol for the EPR experiment

- Alice prepares multiple copies of a bipartite state over $A_i B_i$

- Alice sends parts B_i to Bob

- Bob asks Alice to perform a specific measurement on all A_i



- Alice makes the measurement on A_i and announces the results

- Bob does tomography to verify the expected conditional states

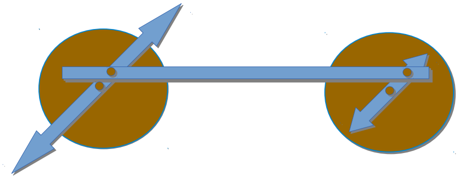
‘Noisy’ EPR steering

The Werner state

$$W_p = p\,|\psi_-\rangle\langle\psi_-| + (1-p)\frac{\mathbb{I}}{2}\otimes\frac{\mathbb{I}}{2}$$

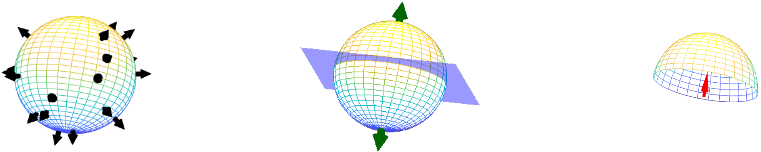
☛ If Alice measures $\hat{n}$, B is ‘collapsed’ to

$$p\,|\hat{n},+\rangle\langle\hat{n},+| + (1-p)\frac{\mathbb{I}}{2} \text{ or } p\,|\hat{n},-\rangle\langle\hat{n},-| + (1-p)\frac{\mathbb{I}}{2}$$



Steering of Werner state: a cheating strategy

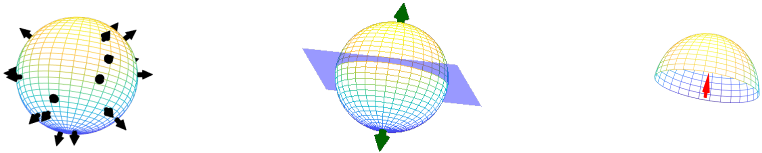
- Alice sends Bob random states B_i on the Bloch sphere
- Bob asks Alice to perform a specific measurement $\hat{n}$ on all A_i
- Alice announces the outcomes $|\hat{n}, \pm\rangle$ for B_i by partitioning the Bloch 'sphere'



- Bob classifies B_i into different outcomes and do tomography to verify that B_i are in the expected states

We say $W_{\frac{1}{2}}$ is unsteerable with projective measurements!

Unsteerable and steerable states



A state ρ is unsteerable from Alice's side if:

- there exists an ensemble of Local Hidden States (LHS) $u(P)$ on the Bloch sphere
- for any measurement $E = \{E_i\}_{i=1}^n$ on A, there exist response functions $0 \leq G_i(P) \leq 1$, $\sum_{i=1}^n G_i(P) = 1$, such that

$$E_i' = \int \mathrm{d}\, S(P) u(P) G_i(P) P$$

where $E_i' = \mathrm{Tr}_A[\rho(E_i \otimes \mathbb{I}_B)]$.

Wiseman et al. PRL '07

The central question

Given a state ρ, is it steerable or unsteerable?

...unsolved even for the simplest case of the two-qubit Werner state!

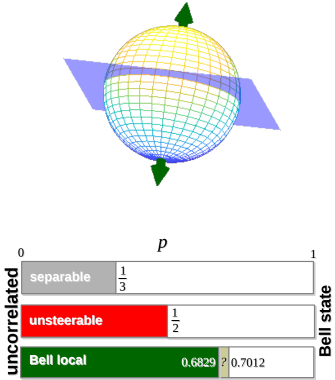
We do understand well:

- ☞ finite number of measurements
- ☞ projective measurements

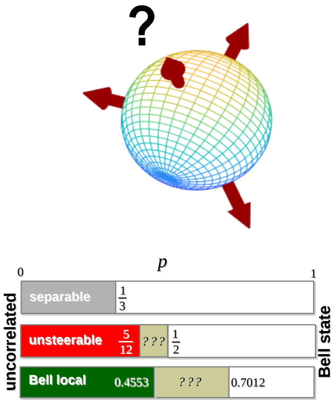
See: Open quantum problem 39 (IQOQI Vienna)

What is the difficulty?

For two-qubit Werner state $W_{\frac{1}{2}}$:



...*with PVMs*



...*with POVMs*

Fact: *For two-qubit states, considering 4-POVM is enough!*

D'Ariano et al 2006; Barrett 2002, Werner 2014, Quintino et al. 2015

Outlines

1. **Steerability as a nesting problem**
2. **The first nesting criterion: nesting by duality**
   - Evidence for unsteerability of $W_{\frac{1}{2}}$ with 4-POVMs
3. **The second nesting criterion: nesting by topology**
   - Further evidence for unsteerability of $W_{\frac{1}{2}}$ with 4-POVMs
   - Proof of unsteerability of $W_{\frac{1}{2}}$ with 3-POVMs
   - Some remarks on the steerability of two-qubit states with 2-POVMs

Steerability as a nesting problem

A state ρ is unsteerable from Alice's side if:

- ☞ there exists an ensemble of Local Hidden States (LHS) $u(P)$ on the Bloch sphere
- ☞ for any measurement $E = \{E_i\}_{i=1}^n$ on A, there exist response functions $0 \leq G_i(P) \leq 1$, $\sum_{i=1}^n G_i(P) = 1$, such that

$$E_i' = \int \mathrm{d}\, S(P) u(P) G_i(P) P$$

where $E_i' = \mathrm{Tr}_A[\rho(E_i \otimes \mathbb{I}_B)]$.

A state ρ is unsteerable with n-POVMs by LHS ensemble u iff

$$(\mathcal{M}^n)' \subseteq \mathcal{K}^n(u)$$

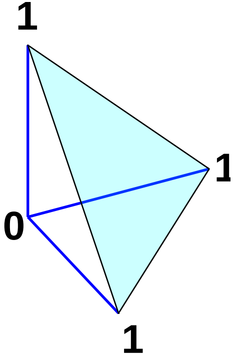
- ☞ $\mathcal{M}^n$: the set of POVMs of n outcomes
- ☞ $(\mathcal{M}^n)'$: the n- steering assemblage (all ensembles Alice can steer)
- ☞ $\mathcal{K}^n(u)$: the n-capacity of u (all ensembles Alice can simulate)

CN, AM, TV & SJ, arXiv:1706.08166

The set of POVMs $\mathcal{M}^n$

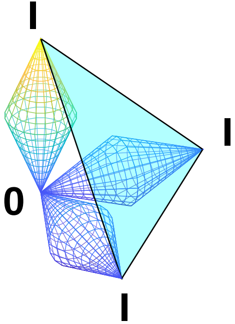
n-probability simplex S^n

$$p_1 \oplus p_2 \oplus \cdots \oplus p_n$$
$$\sum_{i=1}^{n} p_i = 1, 0 \leq p_i \leq 1$$



n-POVM 'simplex' $\mathcal{M}^n$

$$E_1 \oplus E_2 \oplus \cdots \oplus E_n$$
$$\sum_{i=1}^{n} E_i = \mathbb{I}, 0 \leq E_i \leq \mathbb{I}$$



For qubits: the double cone of $0 \leq X \leq \mathbb{I}$

For qubit: with $\{\sigma_i\}_{i=0}^3 = \{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}$

$$X = \frac{1}{2}\sum_{i=0}^{3} x_i \sigma_i$$

Forward cone:

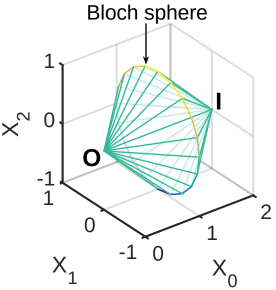
$$0 \leq X : x_0^2 \leq x_1^2 + x_2^2 + x_3^2, 0 \leq x_0$$

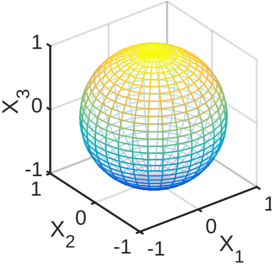
Backward cone:

$$X \leq \mathbb{I} : (2 - x_0)^2 \leq x_1^2 + x_2^2 + x_3^2, x_0 \leq 2$$

Bloch sphere:

$$X \leq \mathbb{I} : x_0^2 = x_1^2 + x_2^2 + x_3^2, x_0 = 1$$

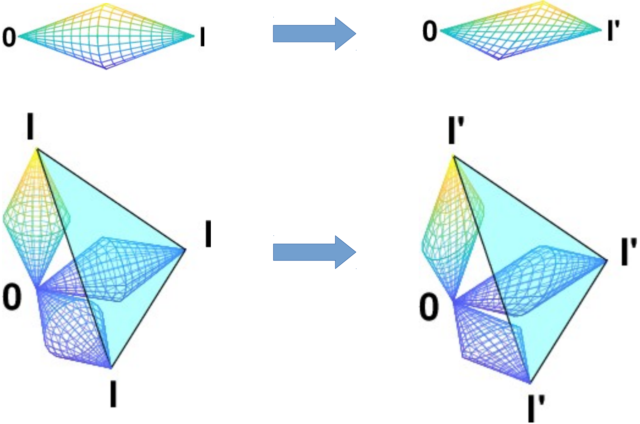




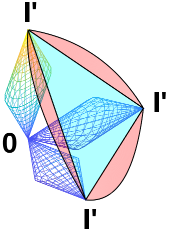
CN & TV, PRA 2016

The steering assemblage $(\mathcal{M}^n)'$ of the POVM 'simplex'

$$\begin{array}{ccc} \text{Alice's system} & \rightarrow & \text{Bob's system} \\ E_1 \oplus E_2 \oplus E_3 & \rightarrow & E_1' \oplus E_2' \oplus E_3' \end{array}$$



The capacity of a distribution $\mathcal{K}^n(u)$



The n-capacity $\mathcal{K}^n(u)$ consists of $K_1 \oplus K_2 \oplus \cdots \oplus K_n$ with

$$K_i = \int \mathrm{d}\, S(P) u(P) G_i(P) P$$

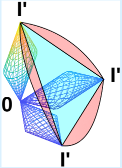
for all possible choices of $0 \leq G_i(P) \leq 1, \sum_{i=1}^{n} G_i(P) = 1$.

CN, AM, TV & SJ, arXiv:1706.08166

Steerability as a nesting problem

A state ρ *is unsteerable with* n*-POVMs by LHS ensemble* u *iff*

$$(\mathcal{M}^n)' \subseteq \mathcal{K}^n(u)$$



- $\mathcal{M}^n$: the set of POVMs of n outcomes
- $(\mathcal{M}^n)'$: the n- steering assemblage (all ensembles Alice can steer)
- $\mathcal{K}^n(u)$: the n-capacity of u (all ensembles Alice can simulate)

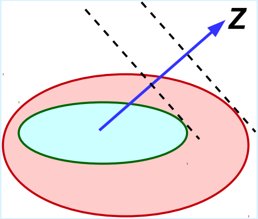
CN, AM, TV & SJ, arXiv:1706.08166

The first criterion: nesting by duality

Let $\mathcal{X}$ and $\mathcal{Y}$ be two non-empty compact convex sets, then $\mathcal{Y} \subseteq \mathcal{X}$ iff

$$\max_{X \in \mathcal{X}} \langle Z, X \rangle \geq \max_{Y \in \mathcal{Y}} \langle Z, Y \rangle$$

for all directions Z.



☞ For $\mathcal{X} = \mathcal{K}^n(u)$ and $\mathcal{Y} = (\mathcal{M}^n)'$, define the gap function

$$\Delta = \min_{Z} \left\{ \max_{K \in \mathcal{K}^n(u)} \langle Z, K \rangle - \max_{E \in \mathcal{M}^n} \langle Z, E' \rangle \right\}$$

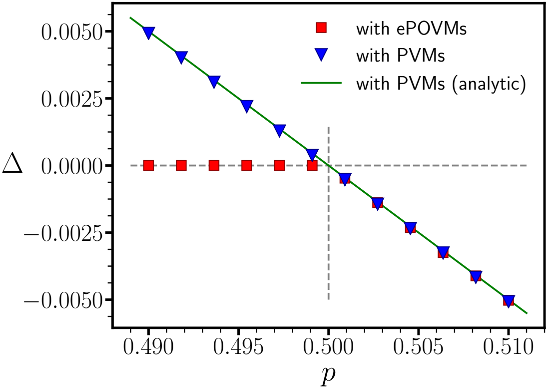
Then $(\mathcal{M}^n)' \subseteq \mathcal{K}^n(u)$ if and only if $\Delta \geq 0$.

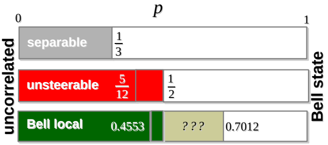
CN, AM, TV & SJ, arXiv:1706.08166

Application: steerability of Werner state with 4-POVMs

$$\Delta = \min_{Z,E} \left\{ \frac{1}{4\pi} \int \mathrm{d}\, S(P) \max_i \langle Z_i, P \rangle - \sum_{i=1}^{4} \mathrm{Tr}[\rho(Z_i \otimes E_i)] \right\}$$

simulated annealing



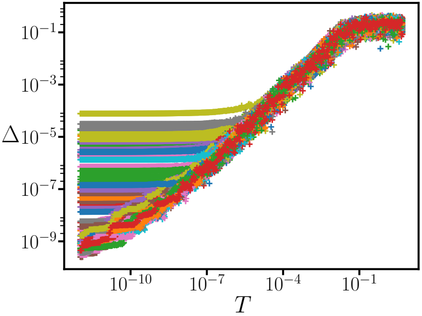


limitation: only heuristic, the region $\frac{1}{2} - 10^{-3} \le p$ *cannot be resolved!*

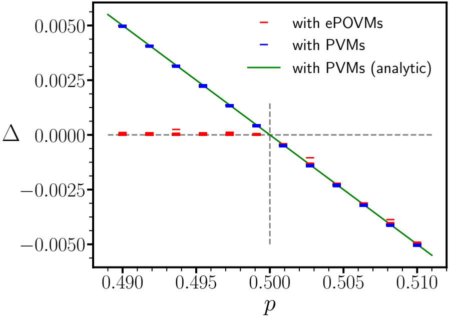
CN, AM, TV & SJ, arXiv:1706.08166

Details of the simulated annealing algorithm

the simulated annealing progress



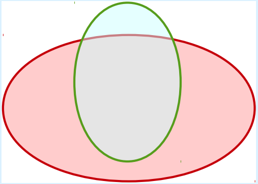
result of 512 replicas



CN, AM, TV & SJ, arXiv:1706.08166

The second criterion: nesting by topology

Let $\mathcal{X}$ *and* $\mathcal{Y}$ *be two non-empty compact convex sets, if* $\mathcal{Y} \subseteq \operatorname{aff} \mathcal{X}$, $\operatorname{int}_r \mathcal{Y} \cap \mathcal{X} \neq \emptyset$ *and* $\partial_r \mathcal{X} \cap \operatorname{int}_r \mathcal{Y} = \emptyset$ *then* $\mathcal{Y} \subseteq \mathcal{X}$.



☞ For $\mathcal{X} = \mathcal{K}^n(u)$ and $\mathcal{Y} = (\mathcal{M}^n)'$, then $(\mathcal{M}^n)' \subseteq \mathcal{K}^n(u)$ if and only if

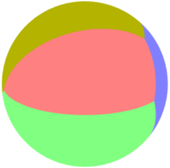
$$\partial_r \mathcal{K}^n(u) \cap \operatorname{int}_r (\mathcal{M}^n)' = \emptyset$$

CN, AM, TV & SJ, *in preparation*

Application: steerability of Werner state with 4-POVMs

The boundary of the capacity parametrised by $Z = \oplus_{i=1}^{4} Z_i$

$$\bar{K}_i(Z) = \frac{1}{4\pi} \int \mathrm{d}\, S(P) \Theta(\langle Z_i, P \rangle - \max_i \langle Z_i, P \rangle) P$$



The relative interior of the steering assemblage

A composite operator $X = \oplus_{i=1}^{4} X_i$ is outside the interior of the steering assemblage of the Werner state if some X_i is outside the interior of the steering image of the positive cone, or

$$\frac{\sqrt{\mathrm{Tr}(X_i^2) - \mathrm{Tr}^2(X_i)}}{\mathrm{Tr}(X_i)} \geq p$$

for some i.

CN, AM, TV & SJ, *in preparation*

A geometric constant

☞*For 4 arbitrary operators* Z_i*, one divides the Bloch sphere into 4 parts* $\mathscr{C}_i$*, each containing projections* P *such that* $\langle Z_i, P\rangle \geq \langle Z_j, P\rangle$ *for* $j \neq i$*.*

☞*Define a geometric constant by:*

$$c_0 = \min_Z \max_i \left\{ \frac{\sqrt{\mathrm{Tr}(\bar{K}_i^2) - \mathrm{Tr}^2(\bar{K}_i)}}{\mathrm{Tr}(\bar{K}_i)} \right\}$$

where $\bar{K}_i = \int_{\mathscr{C}_i} \mathrm{d}\, S(P) P$.

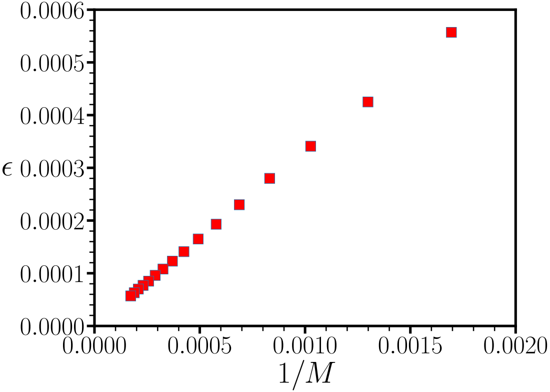
Then the Werner state W_p is unsteerable if and only if $p \leq c_0$!

Conjecture: $c_0 = \frac{1}{2}$

CN, AM, TV & SJ, *in preparation*

Computation of the geometric constant

simulated annealing



with $\epsilon = \frac{1}{2} - c_0$
and M: number of grid points for spherical integration

CN, AM, TV & SJ, *in preparation*

The case of 3-POVMs

3-POVMs are planar!

For a POVM $E = \oplus_{i=1}^{3} E_i$, and $E_i \propto \begin{pmatrix} 1 \\ \boldsymbol{n}_i \end{pmatrix}$, then $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$ are on the same plane, say Oxy.

Planar capacity $\mathcal{K}_z^3(u)$

Response functions $G(\boldsymbol{n})$ are independent of altitude, thus

$$K_i = \frac{1}{4\pi} \int \mathrm{d}\, s(\boldsymbol{a}) g_i(\boldsymbol{a}) \begin{pmatrix} 2 \\ \frac{\pi}{2}\boldsymbol{a} \\ 0 \end{pmatrix}$$

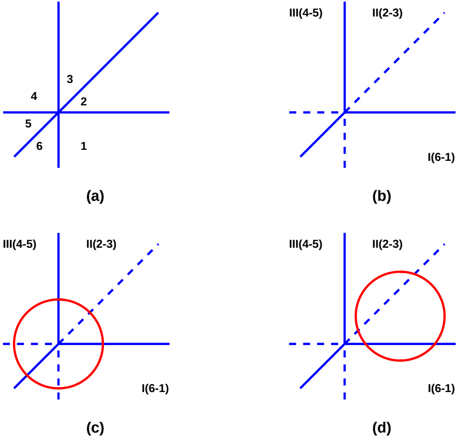
where $\boldsymbol{a}$ is on the unit circle of Oxy.

Werner, JPA (2014); CN, AM, TV & SJ, *in preparation*

The case of 3-POVMs

The boundary of $\mathcal{K}_z^3(u)$

$$\bar{K}_i = \frac{1}{4\pi}\int \mathrm{d}\, s(\boldsymbol{a})\Theta(z_0^i + \boldsymbol{z}^i\cdot\boldsymbol{a})\begin{pmatrix} 2 \\ \frac{\pi}{2}\boldsymbol{a} \\ 0 \end{pmatrix}$$



The case of 2-POVMs

The set of 2-POVM reduces to

$$\mathcal{M} = \{X | 0 \leq X \leq \mathbb{I}\}$$

The steering assemblage reduces to

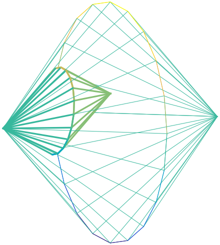
$$\mathcal{M}' = \mathrm{Tr}_A[\rho(\mathcal{M} \otimes \mathbb{I})]$$

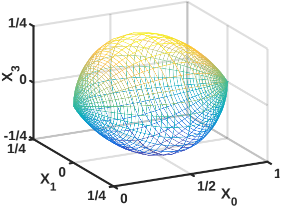
The 2-capacity reduces to

$$\mathcal{K}(u) = \left\{ \int \mathrm{d}\, S(P) u(P) g(P) P \middle| 0 \leq g(P) \leq 1 \right\}$$

For qubit and uniform distribution

$$\partial\mathcal{K}(u) : x_1^2 + x_2^2 + x_3^2 = (1 - x_0)^2 x_0^2$$

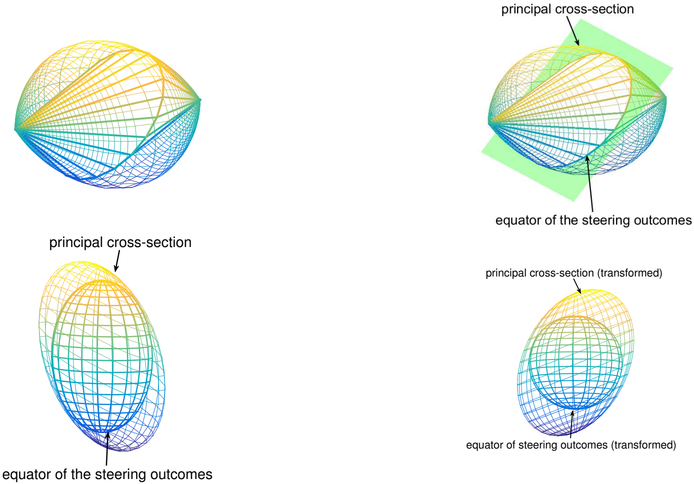




CN & TV, PRA ’16

The case of 2-POVMs

Simplified nesting criterion



☞ Define $r(u)$ to be the inscribed radius of the transformed principal cross-section of $\mathcal{K}(u)$ then ρ is unsteerable iff $r(u) \geq 1$.

See: Jevtic et al., JOSA B '15; CN & TV, EPL '16

Concluding remarks

Quantum steering is stated as a nesting problem of convex objects:

- ☞ Two testing criterions were stated
- ☞ The steerability of the two-qubit Werner state is tested

Future projects:

- ☞ Steerability of other two-qubit states
- ☞ Optimising the LHS ensemble
- ☞ Higher dimensional systems: are PVMs and POVMs equivalent?

Acknowledgement

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Thank you for your attention!