Quantifying quantum incompatibility

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Quantum Incompatibility 2017,
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Outlook

Quantum measurements

Quantum measurements: Instruments

- physical system $\sim \mathscr{H}$, states: $\mathscr{S}(\mathscr{H})$
- measurement outcomes: (Ω, Σ)
- post-measurement system $\sim \mathscr{K}$, trace class: $\mathscr{T}(\mathscr{K})$

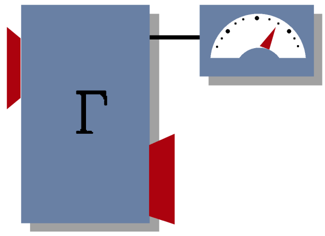
measurement $\sim$ instrument Γ:

- $\Gamma : \Sigma \times \mathscr{S}(\mathscr{H}) \to \mathscr{T}(\mathscr{K})$
- $\Gamma(X, \cdot)$ CP maps, $X = \Omega$: CPTP map
- $\mathrm{tr}[\Gamma(\cdot, \rho)]$ probability measure for all $\rho \in \mathscr{S}(\mathscr{H})$
- denote $\Gamma(X, \cdot) = \Gamma_X$

Parts of a measurement: observable and channel

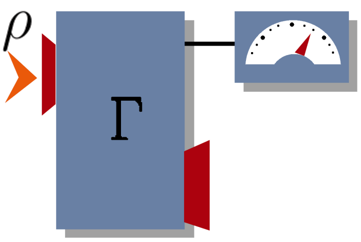
Measurement (instrument Γ) contains a statistics branch, observable M_Γ, and an unconditioned state transformation branch, channel $\mathscr{E}_\Gamma$.

Measurement and its parts



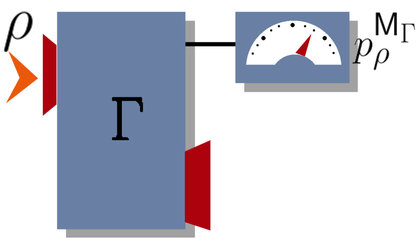
Measurement device $\sim$ instrument Γ
.

Measurement and its parts



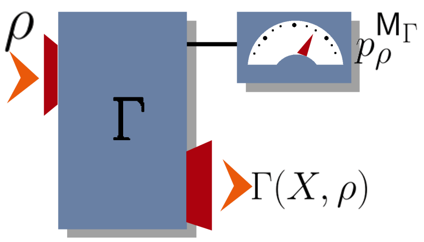
An initial state is subjected to a measurement
. . .

Measurement and its parts



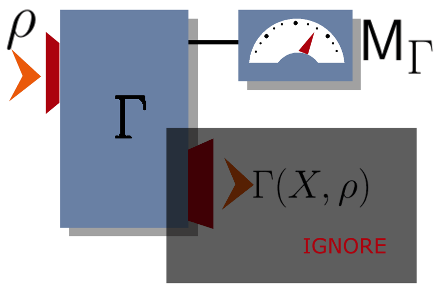
. . . and outcome $\omega \in X$ is registered with prob. $p_\rho^{\mathsf{M}_\Gamma}(X) = \mathrm{tr}[\Gamma(X,\rho)]$. . .

Measurement and its parts



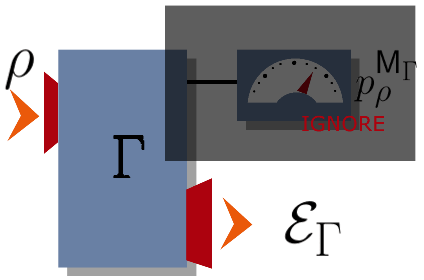
...and, conditioned by this, conditional output state exits the device.

Measurement and its parts

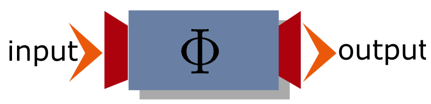


Neglect the state transformations: you obtain the observable (POVM) M_Γ measured by Γ.

Measurement and its parts



Neglect the outcome statistics: you obtain the unconditioned channel (CPTP map) induced by Γ.



From now on, we call elements of quantum measurements, observables, channels, and instruments, as quantum (measurement) devices.

Compatibility and incompatibility

An observable-channel pair $(\mathsf{M}, \mathscr{E})$ is compatible if they can be implemented simultaneously in a measurement.
This means that there is an instrument Γ such that

$$\mathsf{M} = \mathsf{M}_\Gamma, \qquad \mathscr{E} = \mathscr{E}_\Gamma.$$

Let's generalize this notion for other quantum devices (n-tuples of them, $n \geq 2$).

Some notations

Following notations shall be used throughout the rest of this talk.
Fix

- (convex) sets $\mathbf{Q}_j$, $j = 1, \ldots, n$, of similar quantum devices with input state space $\mathscr{S}_{\rm in}$ and output state spaces $\mathscr{S}_{\rm out}^j$,
- $\mathbf{Q} := \mathbf{Q}_1 \times \cdots \times \mathbf{Q}_n$, and
- $\mathscr{S}_{\rm out} := \bigotimes_{j=1}^n \mathscr{S}_{\rm out}^j$, the set $\mathbf{Q}_{\rm joint}$ of devices $\Psi : \mathscr{S}_{\rm in} \to \mathscr{S}_{\rm out}$.

Typically input system is a quantum system: $\mathscr{S}_{\rm in} = \mathscr{S}(\mathscr{H})$.
Output system determines the type of devices studied:

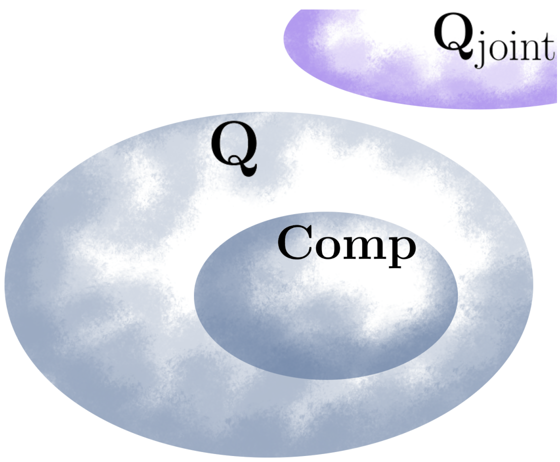
- $\mathscr{S}_{\rm out}^j = \mathscr{S}(\mathscr{K}_j) \Rightarrow \mathbf{Q}_j$ consists of channels.
- $\mathscr{S}_{\rm out}^j$ is a set of probability measures on a measurable space $\Rightarrow \mathbf{Q_j}$ consists of observables.

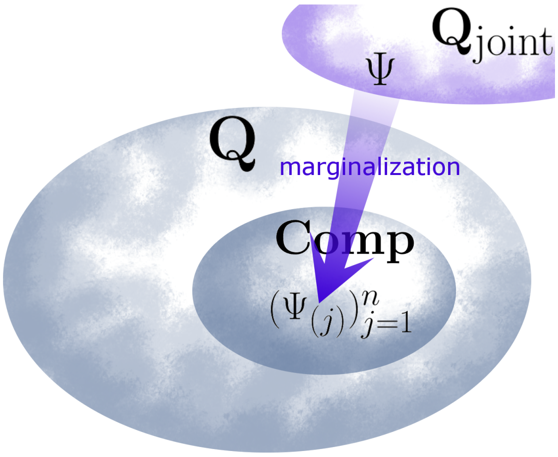
Formal definition

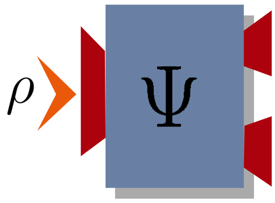
[E. H., T. Heinosaari, J.-P. Pellonpää, *Rev. Math. Phys.* **26**, 1450002 (2014)]:

Definition
Collection $\vec{\Phi} \in \mathbf{Q}$ is compatible if there is a device $\Psi \in \mathbf{Q}_{\rm joint}$ such that $\Phi_j = \Psi_{(j)} = \pi_j \circ \Psi$, $j = 1, \ldots, n$. Otherwise, $\vec{\Phi}$ is incompatible. The subset of compatible device n-tuples $\vec{\Phi} \in \mathbf{Q}$ is denoted by $\mathbf{Comp}$.

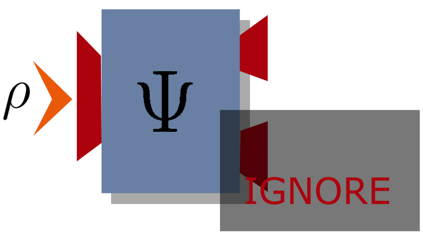
Above, π_j is the j:th marginalization; partial trace, summing up classical outcomes. . .



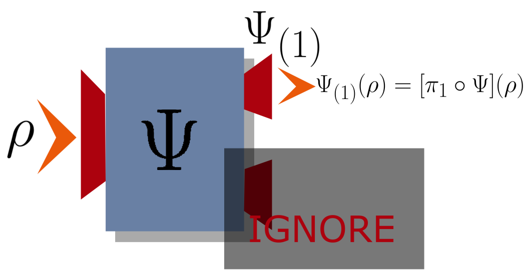




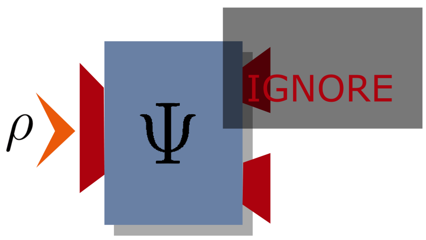
Device Ψ has a joint system as its output system.



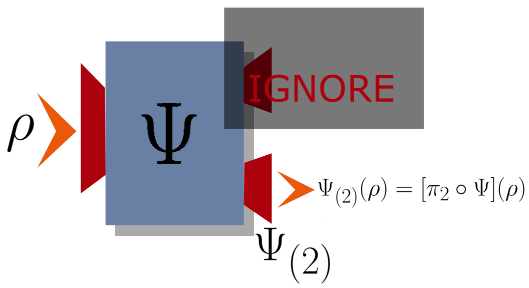
Ignoring the second arm. . .



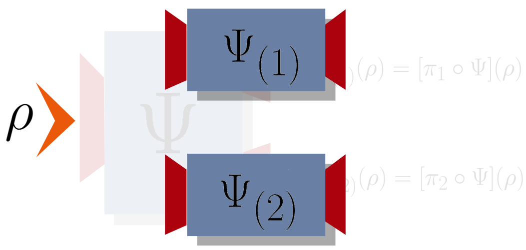
Ignoring the second arm gives the first marginal $\Psi_{(1)}$.



Similarly, one obtains. . .



Similarly, one obtains the second marginal $\Psi_{(2)}$.



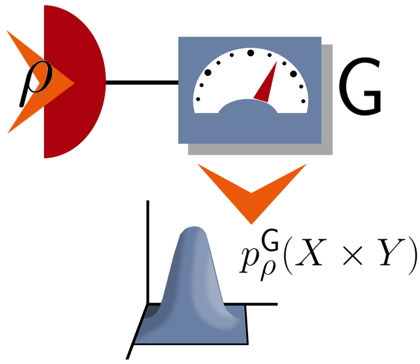
Device Γ is a joint device for the subdevices $\Psi_{(1)}$ and $\Psi_{(2)}$.

Different notions of compatibility

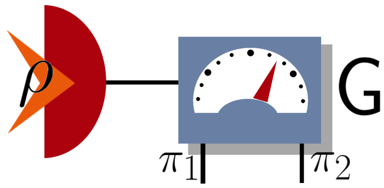
In addition to the observable-channel compatibility, the above definition of compatibility encompasses the following notions for pairs devices of the same type:

- observable-observable case: joint measurability
- channel-channel case: broadcastability

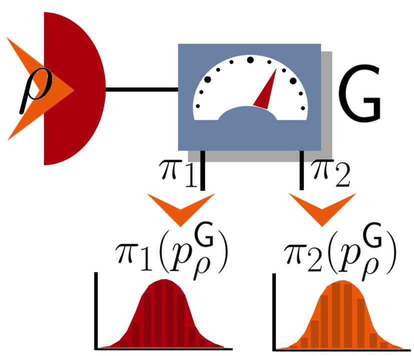
Joint measurability



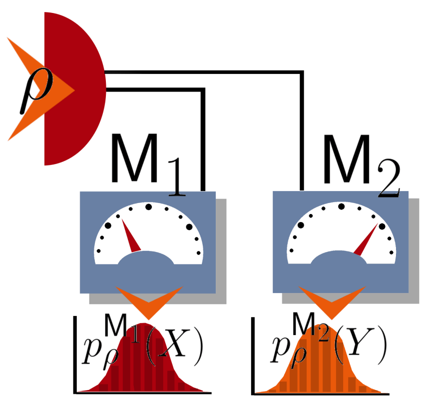
Joint measurability



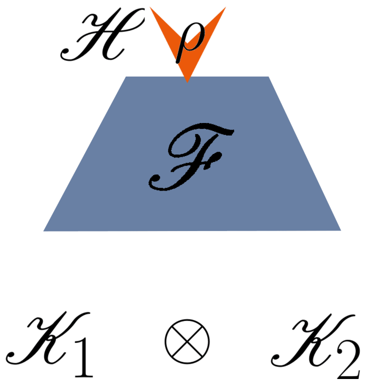
Joint measurability



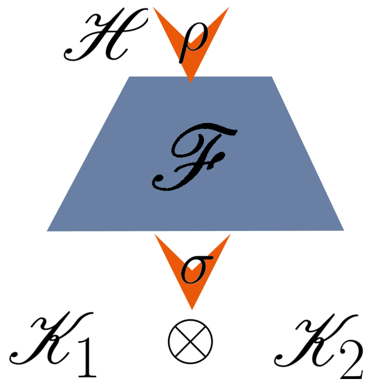
Joint measurability



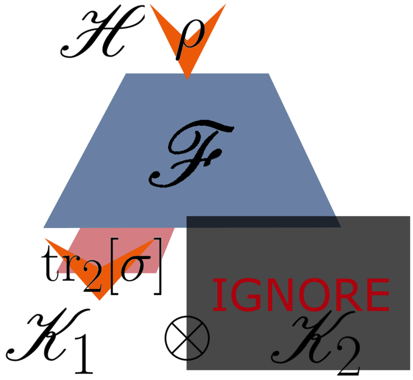
Broadcastability



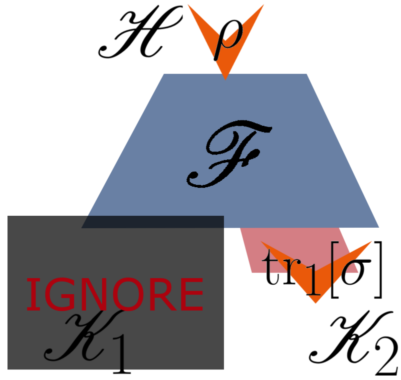
Broadcastability



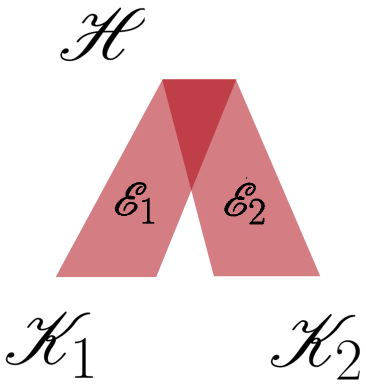
Broadcastability



Broadcastability



Broadcastability



Incompatibility as a resource

Incompatibility and EPR-steering

A bipartite state $\rho_{AB} \in \mathscr{S}(\mathscr{H}_A \otimes \mathscr{H}_B)$ is $(A \to B)$-steerable if there is a collection of observables $\vec{\mathsf{M}} = (\mathsf{M}_1, \ldots, \mathsf{M}_n)$ on subsystem A such that, by measuring $\vec{\mathsf{M}}$ on A, one can steer the conditional sub-system state on B outside a state assemblage like the ones arising from local-hidden-state models.

For this steering to succeed, $\vec{\mathsf{M}}$ has to be incompatible.

- [M. T. Quintino, T. Vértesi, and N. Brunner, *Phys. Rev. Lett.* **113**, 160402 (2014)]
- [R. Uola, T. Moroder, and O. Gühne, *Phys. Rev. Lett.* **113**, 160403 (2014)]

Incompatibility is a resource.

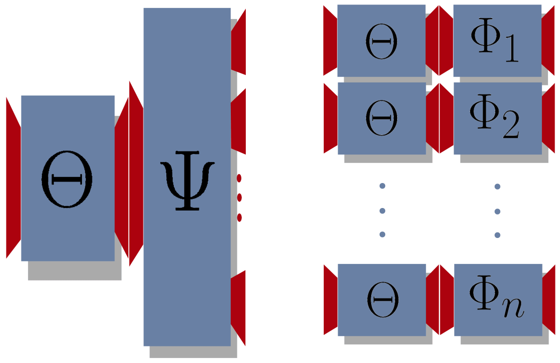
Compatibility non-decreasing operations

Examples on operations on collections of devices that preserve compatibility:

- common pre-processing
- post-processing

Let's illustrate these for a $\vec{\Phi} \in \mathbf{Q}$.

Common pre-processing

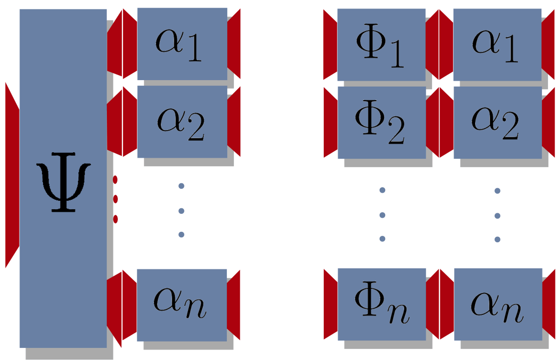


Before applying each device, a common preprocessing (a fixed device $\Theta : \mathscr{S}_{\text{prae}} \to \mathscr{S}_{\text{in}}$) is applied. For a possible joint device Ψ, this means a single preprocessing by Θ.

We denote $\vec{\Phi}' \leq_{\text{prae}} \vec{\Phi}$ if there is Θ such that $\Phi_j' = \Phi_j \circ \Theta$, $j = 1, \ldots, n$.

- If $\vec{\Phi} \in \mathbf{Comp}$, then $\vec{\Phi}' \in \mathbf{Comp}$.
- Typically, the input system of the devices is fully quantum system implying that Θ is a quantum channel.

Post-processing



After operating with device Φ_j, operate with a post-processing (a device $\alpha_j : \mathscr{S}^j_{\text{out}} \to \mathscr{S}^j_{\text{post}}$). For a possible joint device this means post-processing with $\alpha_1 \otimes \cdots \otimes \alpha_n$.

We denote $\vec{\Phi}' \leq_{\text{post}} \vec{\Phi}$ if there is $\vec{\alpha}$ such that $\Phi_j' = \alpha_j \circ \Phi_j$, $j = 1, \ldots, n$.

- If $\vec{\Phi} \in \mathbf{Comp}$, then $\vec{\Phi}' \in \mathbf{Comp}$.
- If, e.g., output system j is classical (Φ_j is an observable), α_j is a classical-to-classical channel, i.e., a statistical operator.

Quantification schemes

Quantification schemes

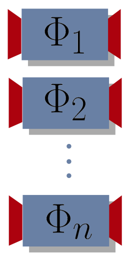
How to measure the separation of $\vec{\Phi} \in \mathbf{Q} \setminus \mathbf{Comp}$ from the (convex) zero-resource set $\mathbf{Comp}$?

Requirements for an incompatibility measure

For $D : \mathbf{Q} \to \mathbf{R}_+$ to be a measure for incompatibility, we require, at least, the following:

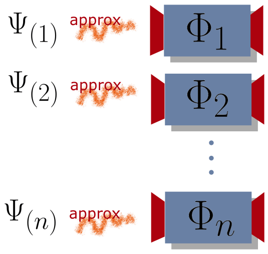
- $D(\vec{\Phi}) = 0$ if and only if $\vec{\Phi} \in \mathbf{Comp}$.
- D is a convex function.
- If $\vec{\Phi}' \leq_{\text{prae}} \vec{\Phi}$, $D(\vec{\Phi}') \leq D(\vec{\Phi})$.
- If $\vec{\Phi}' \leq_{\text{post}} \vec{\Phi}$, $D(\vec{\Phi}') \leq D(\vec{\Phi})$.

Compatible approximations



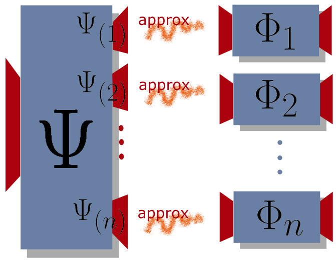
$\vec{\Phi}$ is incompatible.

Compatible approximations



Approximate $\vec{\Phi}$ with. . .

Compatible approximations



Approximate $\vec{\Phi}$ with marginals of a joint device $\Psi : \mathscr{S}_{\text{in}} \to \mathscr{S}_{\text{out}}$.

Determine:

- $\mathbf{Approx}_{\vec{\Phi}}$, the set of all accepted approximate joint devices Ψ for $\vec{\Phi}$,
- a method of evaluating the approximation between Φ_j and $\Psi_{(j)}$, $j = 1, \ldots, n$, for $\Psi \in \mathbf{Approx}_{\vec{\Phi}}$.

Extremize the above approximation over $\Psi \in \mathbf{Approx}_{\vec{\Phi}}$.
$\Rightarrow$ measure for incompatibility

Let us look at two ways of doing this.

Noise robustness of incompatibility

Different types of noise can be mixed with $\vec{\Phi}$:

- compatible noise,

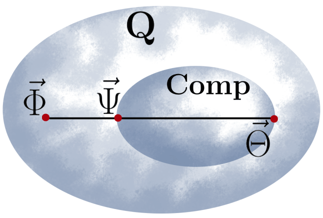
$$\begin{aligned} &\mathbf{Approx}_{\vec{\Phi}} \\ = \;& \{\Psi \,|\, \Psi_{(j)} = w\Phi_j + (1-w)\Theta_j,\ w \in [0,1],\ \vec{\Theta} \in \mathbf{Comp}\} \end{aligned}$$

- general noise,

$$\widetilde{\mathbf{Approx}}_{\vec{\Phi}} = \{\Psi \,|\, \Psi_{(j)} = w\Phi_j + (1-w)\Theta_j,\ w \in [0,1],\ \vec{\Theta} \in \mathbf{Q}\}.$$

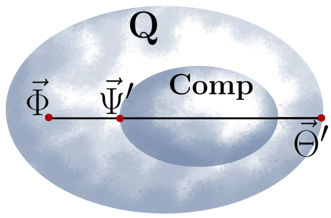
Determine how much noise can be mixed with $\vec{\Phi}$ before the mixture becomes compatible.

$$
\begin{aligned}
r(\vec{\Phi}|\vec{\Theta}) &:= \inf\left\{ s \geq 0 \,|\, \frac{1}{s+1}(\vec{\Phi} + s\vec{\Theta}) \in \mathbf{Comp} \right\}, \\
r(\vec{\Phi}) &:= \inf_{\vec{\Theta} \in \mathbf{Comp}} r(\vec{\Phi}|\vec{\Theta})
\end{aligned}
$$



$\mathbf{Comp} \ni \vec{\Psi} = (\Psi_{(1)}, \ldots, \Psi_{(n)}) = \frac{1}{r(\vec{\Phi})+1}\vec{\Phi} + \frac{r(\vec{\Phi})}{r(\vec{\Phi})+1}\vec{\Theta}$

$$
\begin{aligned}
r(\vec{\Phi}|\vec{\Theta}) &:= \inf\left\{ s \geq 0 \,|\, \frac{1}{s+1}(\vec{\Phi} + s\vec{\Theta}) \in \mathbf{Comp} \right\}, \\
R(\vec{\Phi}) &:= \inf_{\vec{\Theta}' \in \mathbf{Q}} r(\vec{\Phi}|\vec{\Theta}')
\end{aligned}
$$



$\mathbf{Comp} \ni \vec{\Psi}' = (\Psi'_{(1)}, \ldots, \Psi'_{(n)}) = \frac{1}{R(\vec{\Phi})+1}\vec{\Phi} + \frac{R(\vec{\Phi})}{R(\vec{\Phi})+1}\vec{\Theta}'$

Properties of the robustness measures

We call both r and R as robustness of incompatibility. R (and r) has the following properties:

- $0 \leq R(\vec{\Phi}) \leq n-1$ for all $\vec{\Phi} \in \mathbf{Q}$.
- If the input system is of dimension $d < \infty$, $R(\vec{\Phi}) \leq \frac{d(n-1)}{d+n}$.
- The earlier requirements are met.

Reading on this and similar measures:

- E. H., *J. Phys. A: Math. Theor.* **48**, 255303 (2015): this case
- C. Napoli *et al. Phys. Rev. Lett.* **116**, 150502 (2016): robustness of coherence
- G. Vidal and R. Tarrach, *Phys. Rev. A* **59**, 141-155 (1998): robustness of entanglement
- P. Skrzypczyk, M. Navascués, and D. Cavalcanti, *Phys. Rev. Lett.* **112**, 180404 (2014): steerable weight

Utilizing covariance

The input and output systems feature symmetry properties associated with a symmetry group G:

- Group actions:
  - $G \ni g \mapsto \gamma_g \in \mathrm{Aut}(\mathscr{S}_{\mathrm{in}})$,
  - $G \ni g \mapsto \delta_g \in \mathrm{Aut}(\mathscr{S}_{\mathrm{out}})$,
  - $G \ni g \mapsto \delta_g^j \in \mathrm{Aut}(\mathscr{S}_{\mathrm{out}}^j)$, $j = 1, \ldots, n$,
  - $\pi_j \circ \delta_g = \delta_g^j \circ \pi_j$.
- Covariant devices:
  - $\mathbf{Cov}_\gamma^\delta := \{\Psi \in \mathbf{Q}_{\mathrm{joint}} \,|\, \Psi \circ \gamma_g = \delta_g \circ \Psi \ \forall g \in G\}$,
  - $\mathbf{Cov}_\gamma^{\delta^j} := \{\Phi \in \mathbf{Q}_j \,|\, \Phi \circ \gamma_g = \delta_g^j \circ \Phi \ \forall g \in G\}$, $j = 1, \ldots, n$, and
  - $\mathbf{Cov}_\gamma^{\vec{\delta}} := \prod_{j=1}^n \mathbf{Cov}_\gamma^{\delta^j}$.

Utilizing covariance

Under certain conditions, e.g., G is finite, covariance simplifies the evaluation of robustness:

Proposition

Whenever $\vec{\Phi} \in \mathbf{Cov}_\gamma^{\vec{\delta}}$*,*

$$\begin{aligned} r(\vec{\Phi}) &= \inf_{\vec{\Theta} \in \mathbf{Cov}_\gamma^{\vec{\delta}} \cap \mathbf{Comp}} r(\vec{\Phi}|\vec{\Theta}) \\ R(\vec{\Phi}) &= \inf_{\vec{\Theta} \in \mathbf{Cov}_\gamma^{\vec{\delta}}} r(\vec{\Phi}|\vec{\Theta}) \end{aligned}$$

This result greatly simplifies calculating the robustness for physically meaningful sets of devices.

Examples

Let's take a look at three bipartite exemplary cases; an observable-observable, channel-channel, and observable-channel case.

Fourier-coupled rank-1 PVMs

For any finite $d \in \mathbf{N}$, denote by $\mathscr{H}_d$ the d-dimensional complex Hilbert space. Fix

- an orthonormal basis $\{\varphi_j\}_{j \in \mathbf{Z}_d} \subset \mathscr{H}_d$,
- the Fourier transform (unitary operator) $\mathscr{F} \in \mathscr{L}(\mathscr{H}_d)$,

$$\mathscr{F}\varphi_k = \frac{1}{\sqrt{d}} \sum_{j \in \mathbf{Z}_d} e^{i2\pi jk/d} \varphi_j =: \psi_k,$$

- observables $\mathsf{Q} = \big(\mathsf{Q}(j)\big)_{j \in \mathbf{Z}_d}$ and $\mathsf{P} = \big(\mathsf{P}(k)\big)_{k \in \mathbf{Z}_d}$,

$$\mathsf{Q}(j) = |\varphi_j\rangle\langle\varphi_j|, \quad \mathsf{P}(k) = |\psi_k\rangle\langle\psi_k|,$$

  and
- unitary operators $U(q)$, $V(p)$, and $W(q,p)$,

$$U(q)\varphi_j = \varphi_{j+q}, \quad V(p)\varphi_j = e^{i2\pi jp/d}\varphi_j, \quad W(q,p) = U(q)V(p).$$

Fourier-coupled rank-1 PVMs

Weyl covariance largely as in C. Carmeli, T. Heinosaari, and A. Toigo, **Phys. Rev. A 85**, 012109 (2012):

$$W(q,p)^*\mathsf{Q}(j)W(q,p) = \mathsf{Q}(j-q), \quad W(q,p)^*\mathsf{P}(k)W(q,p) = \mathsf{P}(k-p).$$

Using this, one obtains

$$R(\mathsf{Q},\mathsf{P}) = \frac{\sqrt{d}-1}{\sqrt{d}+1}.$$

$\Rightarrow$ as $d \to \infty$, $R(\mathsf{Q},\mathsf{P})$ tends to the maximal value 1.

Maximal robustness for pairs of channels

Let $d < \infty$, $\mathscr{K}_1$ and $\mathscr{K}_2$ be Hilbert spaces, and $\mathscr{E}_1 : \mathscr{S}(\mathscr{H}_d) \to \mathscr{S}(\mathscr{K}_1)$ and $\mathscr{E}_2 : \mathscr{S}(\mathscr{H}_d) \to \mathscr{S}(\mathscr{K}_2)$ be channels.
Using

- monotonicity of R and
- $\mathrm{U}(d)$-symmetry of $\mathrm{id}_{\mathscr{L}(\mathscr{H}_d)}$ (T. Eggeling and R. F. Werner, *Phys. Rev. A* **63**, 042111 (2001)),

one obtains

$$R(\mathscr{E}_1,\mathscr{E}_2) \leq R(\mathrm{id}_{\mathscr{L}(\mathscr{H}_d)},\mathrm{id}_{\mathscr{L}(\mathscr{H}_d)}) = \frac{d-1}{d+1}$$

The same holds for the other measure r.

A rank-1 PVM and a channel

Let $d < \infty$ and $\{\varphi_j\}_{j\in\mathbf{Z}_d} \subset \mathscr{H}_d$ be an orthonormal basis. Define

- the observable $\mathsf{A} = \big(\mathsf{A}(j)\big)_{j\in\mathbf{Z}_d}$,

$$\mathsf{A}(j) = |\varphi_j\rangle\langle\varphi_j|, \quad j \in \mathbf{Z}_d,$$

  and
- the unitaries $W(q,p)$ as earlier.

To evaluate $R(\mathsf{A},\mathrm{id}_{\mathscr{L}(\mathscr{H}_d)})$ it suffices to consider noise of the form $(\mathsf{N},\mathscr{G})$, where

$$\begin{aligned} W(q,p)^*\mathsf{N}(j)W(q,p) &= \mathsf{N}(j-q), \\ \mathscr{G}\big(W(q,p)\rho W(q,p)^*\big) &= W(q,p)\mathscr{G}(\rho)W(q,p)^*. \end{aligned}$$

A rank-1 PVM and a channel

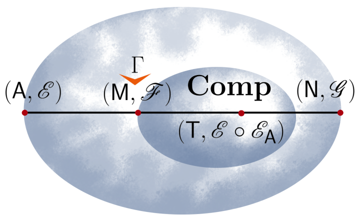
Let $\mathscr{E} : \mathscr{S}(\mathscr{H}_d) \to \mathscr{S}(\mathscr{K})$ be a channel.
Using

- the monotonicity of R and
- the symmetry properties of A and $\mathrm{id}_{\mathscr{L}(\mathscr{H}_d)}$,

one obtains

$$R(\mathsf{A},\mathscr{E}) \leq R(\mathsf{A},\mathrm{id}_{\mathscr{L}(\mathscr{H}_d)}) = \frac{\sqrt{d}-1}{\sqrt{d}+1}.$$

When $\mathrm{id}_{\mathcal{S}(\mathcal{H}_d)} \leq_{\mathrm{post}} \mathcal{E}$:



$$\begin{aligned}
\mathsf{T}(j) &= \frac{1}{d}, \quad \mathcal{E}_{\mathsf{A}}(\rho) = \sum_{j=1}^{d} \mathsf{A}(j)\rho\mathsf{A}(j), \\
\Gamma_j(\rho) &= \frac{\sqrt{d}}{2(\sqrt{d}+1)} \mathcal{E}\big((d^{-1/2}\mathbb{1} + \mathsf{A}(j))\rho(d^{-1/2}\mathbb{1} + \mathsf{A}(j))\big)
\end{aligned}$$

Other interesting examples

There are several other physically motivated cases of incompatible devices where robustness measures could be evaluated:

- Cases where $n \geq 3$, e.g., evaluate $R(\mathrm{id}, \mathsf{Q}, \mathsf{P})$ or $r(\mathrm{id}, \mathsf{Q}, \mathsf{P})$.
- Infinite-dimensional cases, e.g., evaluate $R(\mathsf{Q}, \mathsf{P})$ of $r(\mathsf{Q}, \mathsf{P})$ where (Q, P) is the sharp position-momentum pair on $L^2(\mathbf{R})$.
  - Conjecture:

$$R(\mathsf{Q}, \mathsf{P}) = r(\mathsf{Q}, \mathsf{P}) = 1 \text{ (maximally incompatible)}$$

Other measures for observable incompatibility

Several proposals for measuring observable-observable incompatibility exist:

- $$\begin{aligned}
R_p(\mathsf{M}, \mathsf{N}) &= \inf\{t \geq 0 \,|\, (\mathsf{M}_{t,p}, \mathsf{N}_{t,p}) \in \mathbf{Comp}\} \text{ where} \\
\mathsf{M}_{t,p}(i) &= (1-t)\mathsf{M}(i) + tp(i)\mathbb{1}
\end{aligned}$$
  - P. Busch, T. Heinosaari, J. Schultz, and N. Stevens, *EPL* **103** 10002 (2013)
  - C. Carmeli, T. Heinosaari, and A. Toigo, *Phys. Rev. A* **85**, 012109 (2012)
  - T. Heinosaari, J. Kiukas, and D. Reitzner, *Phys. Rev. A* **92** 022115 (2015)
  - T. Heinosaari, J. Schultz, A. Toigo, and M. Ziman, *Phys. Lett. A* **378** 1695-1699 (2014)
  - Same measure can also be defined for testers (as Mário told us earlier today).

Other measures for observable incompatibility

- $$\begin{aligned}
R_{\mathcal{T}}(\mathsf{M}, \mathsf{N}) &= \inf\{t \geq 0 \,|\, (\mathsf{M}_{t,\mathcal{T}}, \mathsf{N}_{t,\mathcal{T}}) \in \mathbf{Comp}\} \text{ where} \\
\mathsf{M}_{t,\mathcal{T}}(i) &= (1-t)\mathsf{M}(i) + t\mathcal{T}\big(\mathsf{M}(i)\big),
\end{aligned}$$

  and $\mathcal{T}$ is the completely depolarizing channel
  - T. Heinosaari, J. Kiukas, and D. Reitzner, *Phys. Rev. A* **92** 022115 (2015)

Both this measure and the previous one generalize to the multipartite case.

Other measures for observable incompatibility

An entropic measure [A. Barchielli, M. Gregoratti, and A. Toigo: arXiv:1608.01986, arXiv:1705.09949]:

$$c(\vec{\mathsf{M}}) = \inf_{\mathsf{G}} \sup_{\rho} \sum_{j=1}^{n} S(p_\rho^{\mathsf{M}_j} \| p_\rho^{\mathsf{G}_{(j)}}),$$

where G runs through all possible joint observables and $S(\cdot\|\cdot)$ is the Kullback-Leibler distance for probability measures.
A possibility: This measure can be generalized also for collections channels and other quantum devices;

$$c(\vec{\Phi}) = \inf_{\Psi \in \mathbf{Q}_{\rm joint}} \sup_{\rho \in \mathscr{S}_{\rm in}} \sum_{j=1}^{n} D_j\big(\Phi_j(\rho) \| \Psi_{(j)}(\rho)\big)$$