Quantum incompatibility in collective measurements

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Abstract

We study joint measurability of quantum observables in a setting where the experimenter has access to multiple copies of a given quantum system (collective measurements).

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Notations and main concepts

- $\mathcal{H}$ is a finite dimensional Hilbert space of dimension d;
- $\mathsf{A}\colon \Omega \to \mathcal{L}(\mathcal{H})$ denotes a POVM on $\mathcal{H}$ with (finite) outcome space Ω.
- a set of POVMs $\{\mathsf{A}^i\}_{i=1}^n$ is **compatible** if there exists a POVM

$$\mathsf{G}\colon \Omega_1 \times \cdots \times \Omega_n \to \mathcal{L}(\mathcal{H})$$

such that

$$\underbrace{\sum_{x_1 \ldots \hat{x}_j \ldots x_n} \mathsf{G}(x_1, \ldots, x_n)}_{=:\mathsf{G}^{[j]}(x_j)} = \mathsf{A}^j(x_j)$$

Contents

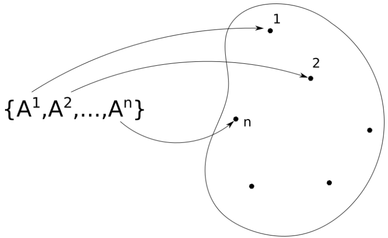
Quantum realization of compatibility structures

Let

$$\mathcal{A} = \{\mathsf{A}^1, \dots \mathsf{A}^n\} \quad (1)$$

be a set of n observables on $\mathcal{H}$.

- construct a diagram with n **vertices** $V = \{1, \dots, n\}$;

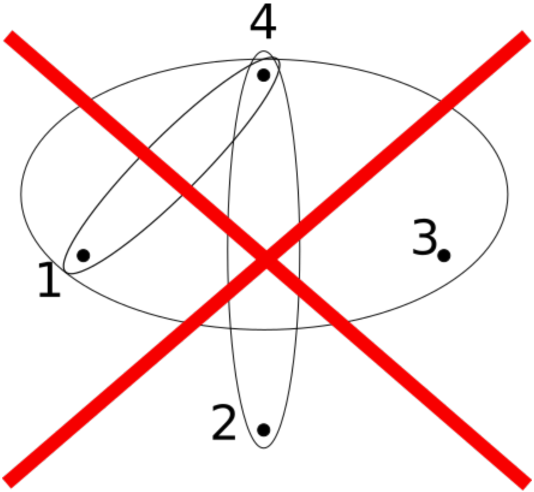


- group together vertices representing compatible subsets of observables: define the set of **edges** $\mathcal{E} \subset 2^V$, as:

$$\{i_1, \dots, i_p\} \in \mathcal{E} \text{ if and only if } \mathsf{A}^{i_1}, \dots, \mathsf{A}^{i_p} \text{ are compatible}$$

Remark

The pair $(V, \mathcal{E})$ is called the **compatibility hypergraph** associated with the set of observables $\mathcal{A}$.



Example

Consider the unsharp spin observables

$$\mathsf{X}_a(\pm 1) \coloneqq \frac{1}{2}\left(\mathrm{id} \pm a\sigma_x\right) \qquad \mathsf{Y}_b(\pm 1) \coloneqq \frac{1}{2}\left(\mathrm{id} \pm b\sigma_y\right)$$

$$\mathsf{Z}_c(\pm 1) \coloneqq \frac{1}{2}\left(\mathrm{id} \pm c\sigma_z\right)$$

We know that

Busch [*Phys. Rev. D* 1986]

The following facts hold.

(1) X_a and Y_b are compatible if and only if $a^2 + b^2 \leqslant 1$.

(2) X_a, Y_b and Z_c are compatible if and only if $a^2 + b^2 + c^2 \leqslant 1$.

Hence we have the following compatibility hypergraphs associated with the noisy spin observables

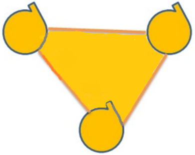


Figure: $a = b = c = \frac{1}{\sqrt{3}}$

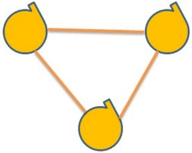


Figure: $a = b = c = \frac{1}{\sqrt{2}}$

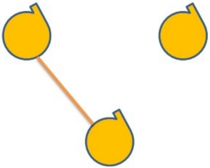


Figure: $a = b = \frac{1}{\sqrt{2}}\ ,\ c = 1$

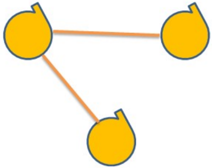


Figure: $a = \frac{1}{\sqrt{2}}\ ,\ b = c = \frac{\sqrt{3}}{2}$

Abstract compatibility hypergraphs

Abstract compatibility hypergraph (a.k.a abstract simplicial complex)

An ***(abstract) compatibility hypergraph*** consists of a set of ***vertices*** V, and a family $\mathcal{E} \subseteq 2^V$ of subsets of V called ***edges*** such that

- all singletons are in $\mathcal{E}$
- $(e \in \mathcal{E} \text{ and } e' \subseteq e) \Rightarrow e' \in \mathcal{E}$

Given an abstract compatibility hypergraph $(V, \mathcal{E})$ we say that it has a ***(quantum) realization***, if it is the associated compatibility hypergraph of a family of observables.

Quantum realization of compatibility hypergraphs

Realizability problem

Given a compatibility hypergraph $(V, \mathcal{E})$, is it associated with a set of POVMs $\{\mathrm{A}^i\}_{i=1}^{|V|}$?

Theorem [Kunjwal,Heunen, Fritz *Phys. Rev. A* **2014**, *89*, 052126]

Every compatibility hypergraph admits a quantum realization.

Idea of the proof

- First prove the result for ***minimal incompatible sets***.

Minimal incompatible sets (MIS)

Sets of incompatible vertices such that each proper subset is compatible.

- reduce any compatibility hypergraph to a **stack** of minimal incompatible sets.

Realizability of MIS

The realizability of any minimal incompatible set is achieved through Clifford algebras.

Proposition

For each $N \in \mathbb{N}$, there exist N hermitian matrices in $M_{2^{\lceil \frac{N}{2} \rceil}}(\mathbb{C})$

$$\Gamma_1 , \dots , \Gamma_N$$

such that

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \, \mathrm{id}$$

Realizability of MIS

For each $N \in \mathbb{N}$, we realize a MIS with N elements using the family of observables

$$\mathsf{A}^k(\pm 1) \coloneqq \frac{1}{2}(\mathrm{id} \pm \eta \Gamma_k) \quad k = 1, \ldots, N$$

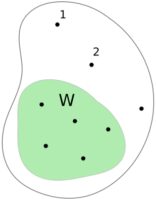
where$0 \leqslant \eta \leqslant 1$.

Proposition

- Any p of the above observables are compatible if and only if $\eta \leqslant \frac{1}{\sqrt{p}}$
- choosing $\frac{1}{\sqrt{N}} \leqslant \eta \leqslant \frac{1}{\sqrt{N-1}}$ one obtains a realization of a MIS with N observables.

Representing a MIS

Let $\mathcal{M}$ be the set of all MIS in $(V, \mathcal{E})$, and let $W \in \mathcal{M}$ be a MIS represented on $\mathcal{H}_W$.



We represent all elements of $V \setminus W$ with trivial observables T:

$$\mathsf{A}^k_W = \begin{cases} \mathsf{A}^k & \text{if } k \in W \\ \mathsf{T} & \text{if } k \notin W \end{cases}$$

Hence, for each V ... graph

The claim is that

$$(V, \tilde{\mathcal{E}}) = (V, \mathcal{E})$$

.

Consider the direc... space

$$\tilde{\mathcal{H}} = \bigoplus_{W \in \mathcal{M}} \mathcal{H}_W \qquad \tilde{\mathsf{A}}^k = \bigoplus_{W \in \mathcal{M}} \mathsf{A}^k_W$$

This family of observables has an associated compatibility hypergraph $(V, \tilde{\mathcal{E}})$.

Reduction to MIS

The proof of the previous remark is based on the following observation

Lemma

Fix $X \subseteq V$. Either $X \in \mathcal{E}$ (i.e. all observables in X are compatible) or there exists a MIS contained in X.

Proof of the Lemma

It is an inductive proof. Fix $x \in X$ and consider all 2 elements sets $\{x, y\} \subseteq X$. There are two possibilities

- there exists one which is incompatible. This is a MIS and we are done
- they are all compatible

If the second possibility occurs, consider all the 3 elements sets. There are two possibilities

- there exists one which is incompatible. This is a MIS and we are done
- they are all compatible

Continue until you fill X.

Contents

k-compatibility

We put ourselves in a framework where we have access to multiple copies of ρ

$$\rho^{\otimes k}$$

k-compatibility

The observables $\{\mathsf{A}^i\}_{i=1}^n$ are k–**compatible** if there exists an observable

$$\mathsf{G} \colon \Omega_1 \times \cdots \times \Omega_n \to \mathcal{L}(\mathcal{H}^{\otimes k})$$

such that, for each i

$$\mathrm{tr}\left[\mathsf{G}^{[i]}(x_i)\rho^{\otimes k}\right] = \mathrm{tr}\left[\mathsf{A}^i(x_i)\rho\right]$$

Properties of k-compatibility

P1 ***Any collection of n observables is n-compatible.***

P2 ***Any subset of a k-compatible set of observables is k-compatible.***

P3 ***Any collection of k-compatible observables is k'-compatible for all $k' \geqslant k$.***

P4 ***If $\mathcal{A}_1$ is k_1-compatible and $\mathcal{A}_2$ is k_2-compatible, then $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is $(k_1 + k_2)$-compatible.***

Abstract compatibility stack

P2 ***Any subset of a k-compatible set of observables is k-compatible.***

P1 ***Any collection of n observables is n-compatible.***

P4 ***If $\mathcal{A}_1$ is k_1-compatible and $\mathcal{A}_2$ is k_2-compatible, then $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is $(k_1 + k_2)$-compatible.***

Abstract compatibility stack

Let V be a finite set with n elements and let $\mathcal{E}_k$ be a set of non-empty subsets of V for $k = 1, \ldots, n$. We denote $H_k = (V, \mathcal{E}_k)$.
The list $(H_1, \ldots, H_n)$ of hypergraphs is a ***compatibility stack*** if

(S1) each $H_k = (V, \mathcal{E}_k)$ is a joint measurability hypergraph, (P2)

(S2) $\mathcal{E}_1$ contains all singleton sets and $\mathcal{E}_n = 2^V$, (P1)

(S3) if $\mathcal{A} \in \mathcal{E}_k$ and $\mathcal{B} \in \mathcal{E}_l$, then $\mathcal{A} \cup \mathcal{B} \in \mathcal{E}_{k+l}$.(P4)

It is natural to define

Index of incompatibility

The **index of incompatibility** of a set of observables

$$\mathcal{A} = \{\mathsf{A}^i\}_{i=1}^n$$

is

$$\mathrm{i}(\mathcal{A}) = \min\{k \in \mathbb{N} \mid \{\mathsf{A}^i\}_{i=1}^n \text{ are } k-\text{compatible}\}$$

Properties

1. $1 \leqslant \mathrm{i}(\mathcal{A}) \leqslant \#\mathcal{A}$;
2. if $\mathcal{A} \subseteq \mathcal{B}$, then $\mathrm{i}(\mathcal{A}) \leqslant \mathrm{i}(\mathcal{B})$;
3. $\mathrm{i}(\mathcal{A}_1 \cup \mathcal{A}_2) \leqslant \mathrm{i}(\mathcal{A}_1) + \mathrm{i}(\mathcal{A}_2)$;
4. $\mathrm{i}(\mathcal{A}) = 1$ if and only if $\mathcal{A}$ is compatible.

Dimension 2

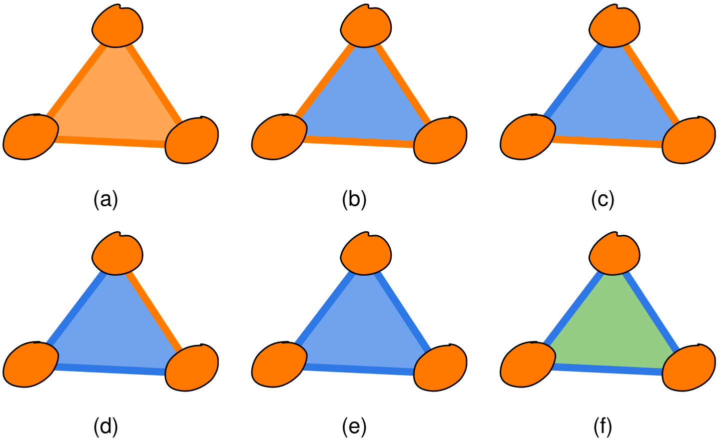
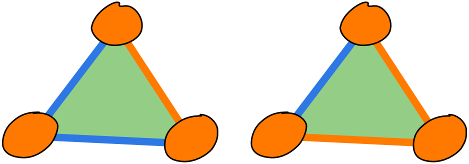


Figure: All possible compatibility stacks with three vertices. Orange color marks index 1, blue marks index 2 and green marks index 3.

Other possibilities are forbidden. For example:



(a) (b)

Figure: Examples of impossible k-compatibility relations for three observable. Orange color marks index 1, blue marks index 2 and green marks index 3.

Playing with stacks: dimension 3

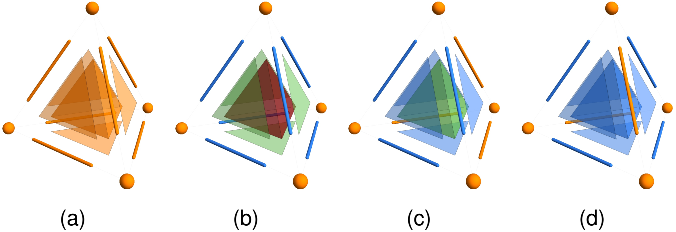


Figure: Compatibility stacks with four vertices can be represented by colored tetrahedrons. As before, orange color marks index 1, blue marks index 2 and green marks index 3. In addition, index 4 is marked by red color. Case (a) represents the most compatible case, where we need only a single copy of a state to measure all four observables, whereas the case (b) is in this respect the worst one, as we need a new copy of a state for each measurement. Cases (c) and (d) show some intermediate possibilitites of compatibility stacks.

Realizability of compatibility stacks in dimension two

- We have 6 compatibility stacks. Let us see what we can say with known results.
- We try to use noisy versions of spin observables X_a , Y_b , Z_c

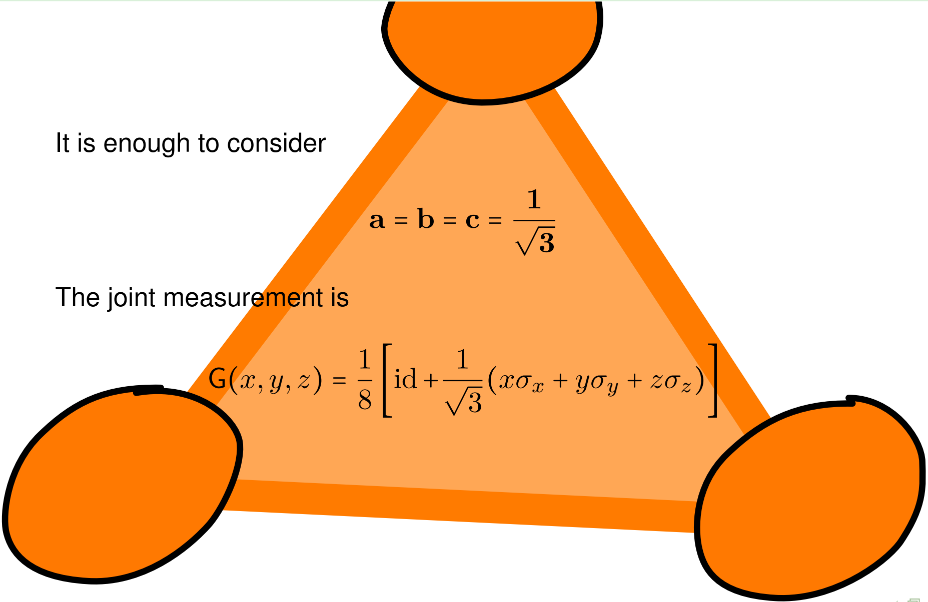
Busch [*Phys. Rev. D* 1986]

The following facts hold.

(1) X_a and Y_b are compatible if and only if $a^2 + b^2 \leqslant 1$.

(2) X_a, Y_b and Z_c are compatible if and only if $a^2 + b^2 + c^2 \leqslant 1$.

First compatibility stack in dimension 2



It is enough to consider

$$\mathbf{a} = \mathbf{b} = \mathbf{c} = \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}$$

The joint measurement is

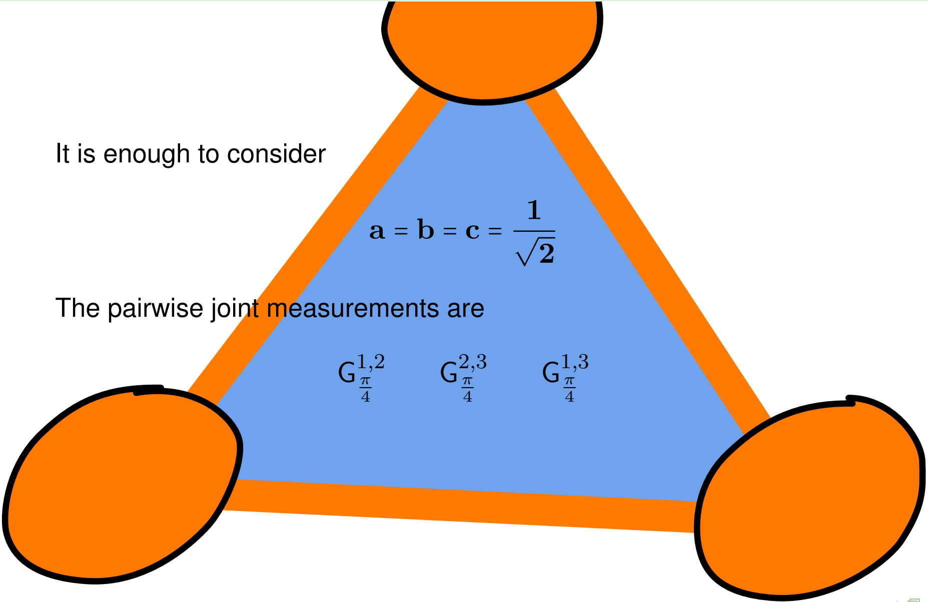
$$\mathsf{G}(x,y,z) = \frac{1}{8}\left[\mathrm{id} + \frac{1}{\sqrt{3}}(x\sigma_x + y\sigma_y + z\sigma_z)\right]$$

The observables

$$\mathsf{G}^{1,2}_{\alpha}(x,y) = \frac{1}{4}(\mathrm{id} + x\sin\alpha\,\sigma_x + y\cos\alpha\,\sigma_y),$$
$$\mathsf{G}^{2,3}_{\beta}(y,z) = \frac{1}{4}(\mathrm{id} + y\sin\beta\,\sigma_y + z\cos\beta\,\sigma_z),$$
$$\mathsf{G}^{1,3}_{\gamma}(x,z) = \frac{1}{4}(\mathrm{id} + x\cos\gamma\,\sigma_x + z\sin\gamma\,\sigma_z).$$

are pairwise joint observables that are on the boundary of the compatibility region of two noisy spin observables.

Second compatibility stack in dimension 2



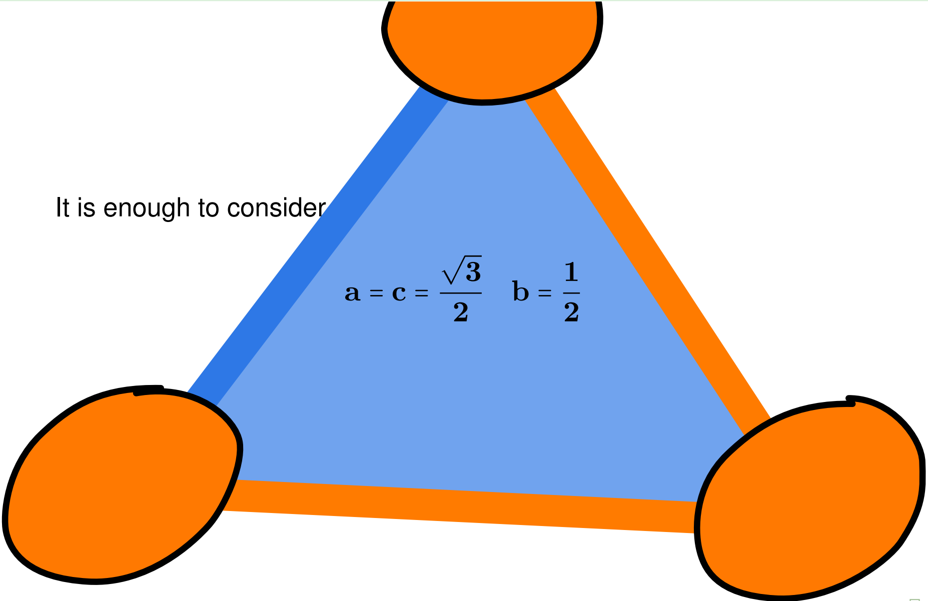
It is enough to consider

$$\mathbf{a} = \mathbf{b} = \mathbf{c} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$$

The pairwise joint measurements are

$$\mathsf{G}^{1,2}_{\frac{\pi}{4}} \qquad \mathsf{G}^{2,3}_{\frac{\pi}{4}} \qquad \mathsf{G}^{1,3}_{\frac{\pi}{4}}$$

Third compatibility stack in dimension 2

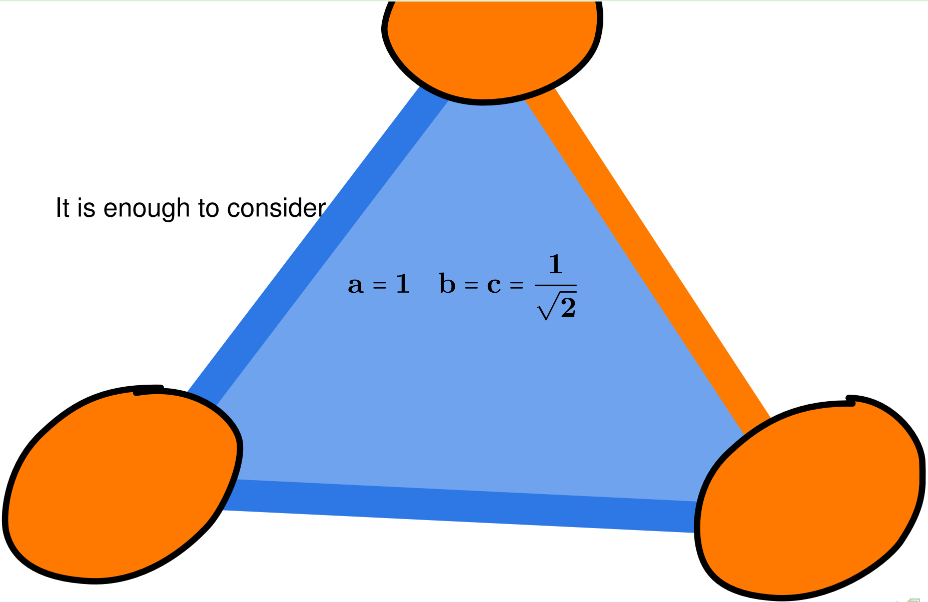


It is enough to consider

$$\mathbf{a} = \mathbf{c} = \frac{\sqrt{3}}{2} \quad \mathbf{b} = \frac{1}{2}$$

Fourth compatibility stack in dimension 2

It is enough to consider



$$\mathbf{a} = 1 \quad \mathbf{b} = \mathbf{c} = \frac{1}{\sqrt{2}}$$

Open cases

There are two diagram left:

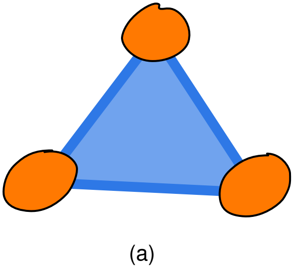


Figure: Open cases

Fifth compatibility stack in dimension 2

We need the following result

Proposition

X_a, Y_b and Z_c are 2-compatible if there are numbers $\lambda_1, \lambda_2, \lambda_3 \in [0,1]$ and $\alpha, \beta, \gamma \in [0, \pi/2]$ such that

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

and

$$\begin{cases} a \leqslant \lambda_1 + \lambda_2 \cos\gamma + \lambda_3 \sin\alpha \\ b \leqslant \lambda_1 \sin\beta + \lambda_2 + \lambda_3 \cos\alpha \\ c \leqslant \lambda_1 \cos\beta + \lambda_2 \sin\gamma + \lambda_3 \end{cases} . \tag{2}$$

Proof

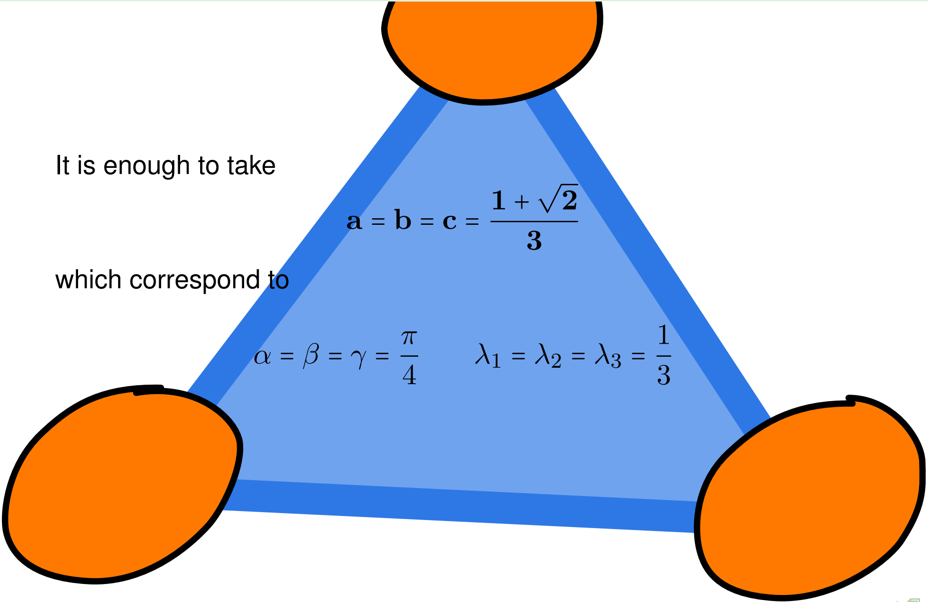
Proof

Consider the $\mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^2)$-valued observable:

$$\mathsf{G}(x,y,z) = \lambda_1 \mathsf{X}(x) \otimes \mathsf{G}^{2,3}_{\beta}(y,z) + \lambda_2 \mathsf{Y}(y) \otimes \mathsf{G}^{1,3}_{\gamma}(x,z) + \lambda_3 \mathsf{Z}(z) \otimes \mathsf{G}^{1,2}_{\alpha}(x,y)$$

It is a a 2-copy joint observable of the three observables X_a, Y_b and Z_c.

Fifth compatibility stack in dimension 2

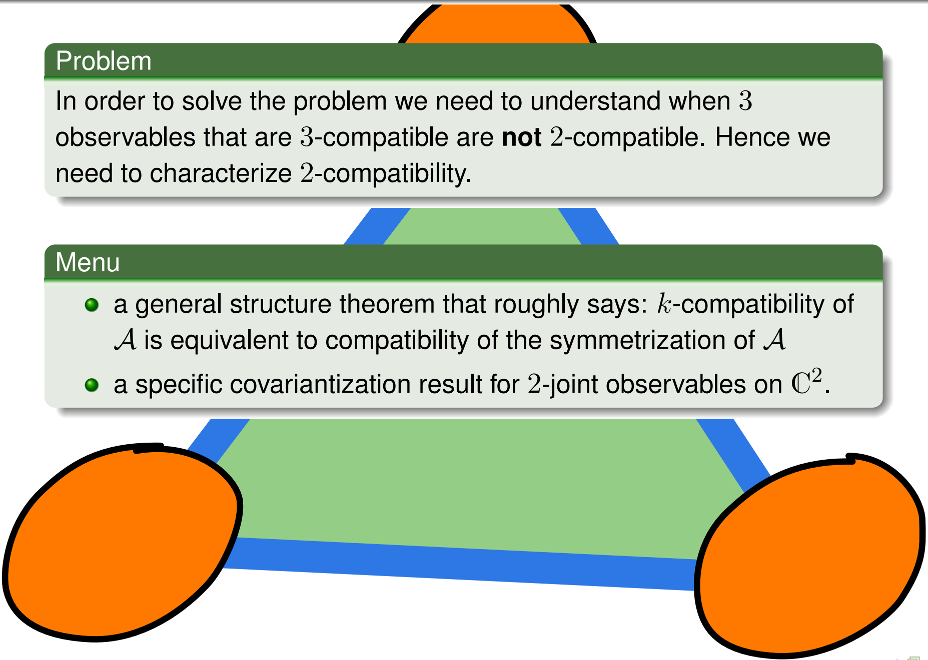


It is enough to take

$$\mathbf{a} = \mathbf{b} = \mathbf{c} = \frac{\mathbf{1} + \sqrt{\mathbf{2}}}{\mathbf{3}}$$

which correspond to

$$\alpha = \beta = \gamma = \frac{\pi}{4} \qquad \lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$$



Problem

In order to solve the problem we need to understand when 3 observables that are 3-compatible are **not** 2-compatible. Hence we need to characterize 2-compatibility.

Menu

- a general structure theorem that roughly says: k-compatibility of $\mathcal{A}$ is equivalent to compatibility of the symmetrization of $\mathcal{A}$
- a specific covariantization result for 2-joint observables on $\mathbb{C}^2$.

Structure theorem: preparation

The symmetric group S_k acts on $\mathcal{H}^{\otimes k}$: if $p \in S_k$ is any permutation, its action is defined as

$$\sigma(p)(\psi_1 \otimes \ldots \otimes \psi_k) = \psi_{p^{-1}(1)} \otimes \ldots \otimes \psi_{p^{-1}(k)} .$$

The symmetrizer channel (Heisenberg picture)

We then define the *symmetrizer channel* Σ_k on $\mathcal{L}(\mathcal{H}^{\otimes k})$ as

$$\Sigma_k(A) = \frac{1}{k!} \sum_{p \in S_k} \sigma(p) A \sigma(p)^* \qquad A \in \mathcal{L}(\mathcal{H}^{\otimes k}) .$$

Σ_k is an orthogonal projection onto the linear subspace $\mathrm{Sym}_k(\mathcal{L}(\mathcal{H}))$ of the k-symmetric tensor operators in $\mathcal{L}(\mathcal{H}^{\otimes k})$.

The *symmetric product* of two operators $A_1 \in \mathrm{Sym}_{k_1}(\mathcal{L}(\mathcal{H}))$ and $A_2 \in \mathrm{Sym}_{k_2}(\mathcal{L}(\mathcal{H}))$ is the operator $A_1 \odot A_2 \in \mathrm{Sym}_{k_1+k_2}(\mathcal{L}(\mathcal{H}))$ with

$$A_1 \odot A_2 = \Sigma_{k_1+k_2}(A_1 \otimes A_2)\,.$$

The symmetric product is associative and commutative.

Structure theorem

Theorem

The $\mathcal{L}(\mathcal{H})$-valued observables $\mathsf{A}_1, \ldots, \mathsf{A}_n$ on $\Omega_1, \ldots, \Omega_n$ are k-compatible if and only if there exists a $\mathcal{L}(\mathcal{H}^{\otimes k})$-valued observable $\widetilde{\mathsf{G}}$ on $\Omega_1 \times \ldots \times \Omega_n$ such that

- $\widetilde{\mathsf{G}}^{[i]}(x) = \mathrm{id}^{\otimes(k-1)} \odot \mathsf{A}_i(x) \qquad \forall i = 1, \ldots, n,\ x \in \Omega_i$
- $\widetilde{\mathsf{G}}(x_1, \ldots, x_n) \in \mathrm{Sym}_k(\mathcal{L}(\mathcal{H}))$ for all $x_1, \ldots, x_n$.

Corollary

The $\mathcal{L}(\mathcal{H})$-valued observables $\mathsf{A}_1, \ldots, \mathsf{A}_n$ are k-compatible if and only if the $\mathcal{L}(\mathcal{H}^{\otimes k})$-valued observables $\widetilde{\mathsf{A}}_1, \ldots, \widetilde{\mathsf{A}}_n$ are compatible, where

$$\widetilde{\mathsf{A}}_i(x) = \mathrm{id}^{\otimes(k-1)} \odot \mathsf{A}_i(x) \,.$$

Example

Let A_1 and A_2 be 2-outcome observables defined by positive operators A_1 and A_2, respectively. That is, $\Omega_1 = \Omega_2 = \{+1, -1\}$, and $\mathsf{A}_i(+1) = A_i$ for $i = 1, 2$.
These are always 2-compatible:

- **obvious choice**:

$$\mathsf{G}(x, y) = \mathsf{A}_1(x) \otimes \mathsf{A}_2(y)$$

- **symmetric choice:**

$$\widetilde{\mathsf{G}}(x, y) = \frac{1}{2}\left[\mathsf{A}_1(x) \otimes \mathsf{A}_2(y) + \mathsf{A}_2(y) \otimes \mathsf{A}_1(x)\right]$$

Explicitly:

$$\widetilde{\mathsf{G}}(+1,+1) = \frac{1}{2}\left(A_1 \otimes A_2 + A_2 \otimes A_1\right) \quad \widetilde{\mathsf{G}}(-1,+1) = \frac{1}{2}\left(A_1^c \otimes A_2 + A_2 \otimes A_1^c\right)$$
$$\widetilde{\mathsf{G}}(+1,-1) = \frac{1}{2}\left(A_1 \otimes A_2^c + A_2^c \otimes A_1\right) \quad \widetilde{\mathsf{G}}(-1,-1) = \frac{1}{2}\left(A_1^c \otimes A_2^c + A_2^c \otimes A_1^c\right)$$

Idea of the proof

The ⇐-direction is a simple calculation. Let us consider ⇒. Since $\mathsf{A}^1, \ldots, \mathsf{A}^n$ are k-compatible, there exists

$$\mathsf{G}\colon \Omega_1 \times \cdot \times \Omega_n \to \mathcal{L}(\mathcal{H}^{\otimes k})$$

such that

$$\operatorname{tr}\left[\mathsf{G}^{[i]}(x)\rho^{\otimes k}\right] = \operatorname{tr}\left[\mathsf{A}^k(x)\rho\right]$$

Define

$$\widetilde{\mathsf{G}} \coloneqq \Sigma_k \circ \mathsf{G}$$

It is clear that it is symmetric.
The condition for k-compatibility is easily checked:

$$\begin{aligned}\operatorname{tr}\left[\widetilde{\mathsf{G}}^{[i]}(x)\rho^{\otimes k}\right] &= \operatorname{tr}\left[\Sigma_k \circ \mathsf{G}^{[i]}(x)\rho^{\otimes k}\right] \\ &= \operatorname{tr}\left[\mathsf{G}^{[i]}(x)\Sigma_k\rho^{\otimes k}\right] \\ &= \operatorname{tr}\left[\mathsf{A}^k(x)\rho\right]\end{aligned}$$

The explicit computation of the margins is more complicated.

We have reduce the problem to the joint measurability of symmetrized version of the original observables. This fact reduces the number of degrees of freedom of a k-joint measurement, but they are still too much.

We need more symmetry to make the problem manageable.

Strategy

Find a result like
"if there exists a k-joint measurements G, then there exists a covariant one $\widehat{\mathsf{G}}$" (**covariantization**)
so that:

$$\widehat{\mathsf{G}}(g.\omega) = U(g)\widehat{\mathsf{G}}(\omega)U(g)^*$$

Key technical result

Proposition

Suppose G is a finite group, Ω is a G-space and U is a unitary representation of G in the Hilbert space $\mathcal{K}$. Let $\mathcal{F}$ be a collection of subsets of Ω such that

$$g.X = \{g.x \mid x \in X\} \in \mathcal{F} \quad \text{for all } g \in G \text{ and } X \in \mathcal{F}\,.$$

Then, for any observable $\mathsf{G} : \Omega \to \mathcal{L}(\mathcal{K})$ satisfying the relation

$$\mathsf{G}(g.X) = U(g)\mathsf{G}(X)U(g)^* \qquad \forall X \in \mathcal{F}\,,\ g \in G\,,$$

the observable $\widehat{\mathsf{G}} : \Omega \to \mathcal{L}(\mathcal{K})$ given by

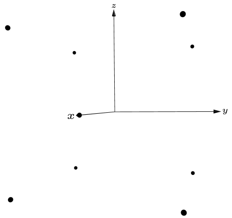
$$\widehat{\mathsf{G}}(x) = \frac{1}{\#G} \sum_{g \in G} U(g)^* \mathsf{G}(g.x) U(g) \qquad \forall x \in \Omega$$

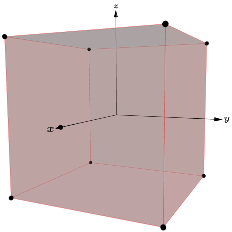
is such that

1. $\widehat{\mathsf{G}}(g.x) = U(g)\widehat{\mathsf{G}}(x)U(g)^*$ for all $x \in \Omega$ and $g \in G$;
2. $\widehat{\mathsf{G}}(X) = \mathsf{G}(X)$ for all $X \in \mathcal{F}$.

We have to look at the geometry of the outcome space

$$\Omega = \{\pm 1\} \times \{\pm 1\} \times \{\pm 1\}$$

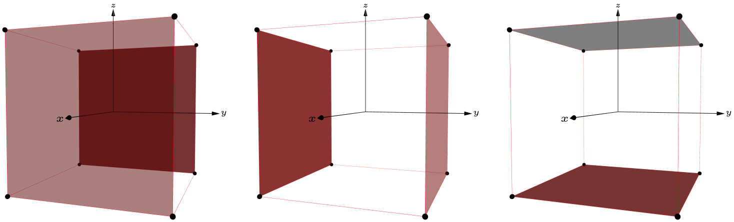




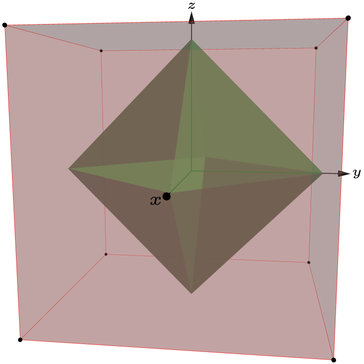
The family $\mathcal{F}$ that we need to "preserve"

There are six sets that must be preserved by any covariantization techinique: the sets giving the six margins:

$$X_\pm = \{(\pm 1, 1, 1)\,, (\pm 1, 1, -1)\,, (\pm 1, -1, -1)\,, (\pm 1, -1, 1)\}$$
$$Y_\pm = \{(1, \pm 1, 1)\,, (1, \pm 1, -1)\,, (-1, \pm 1, -1)\,, (-1, \pm 1, 1)\}$$
$$Z_\pm = \{(1, 1, \pm 1)\,, (1, -1, \pm 1)\,, (-1, 1, \pm 1)\,, (-1, -1, \pm 1)\}$$



The octahedron sits "inside" the outcome space $\Omega = \{\pm 1\}^3$.



Hence the octahedral group acts transitively on Ω. The stabilizer at a given point coincide with the $\frac{2}{3}\pi$-rotations around the corresponding bisetrix.

Octahedral symmetry

It is the 24 elements subgroup O of $SO(3)$ preserving the octahedron

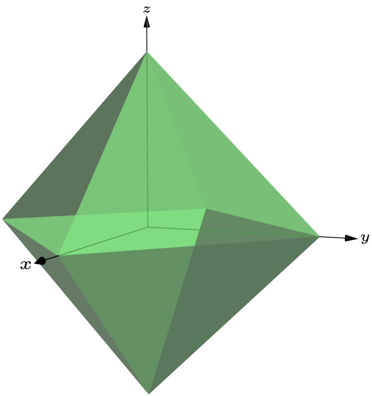


Figure: The octahedral group: rotations preserving the octahedron

Generators of the group O

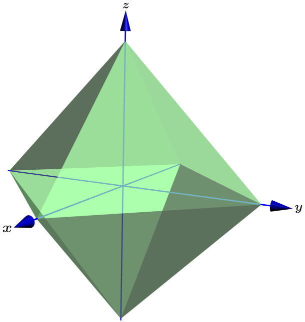


Figure: $\frac{\pi}{2}$-Rotations about coordinate axes

Representation of the Octahedral group O

We take the standard representation π of $SU(2)$ on $\mathbb{C}^2$. The tensor product representation $\pi \otimes \pi$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ descends to a (non projective) representation of $SO(3)$ and, by restriction, to a unitary representation of the Octahedral group.

Classification of symmetric O-covariant observables

Proposition

A map $\mathsf{G} : \Omega \to \mathcal{L}(\mathcal{H}^{\otimes 2})$ is a symmetric and U-covariant observable if and only if there exist real numbers α and β with $\alpha \geqslant 0$, $\beta \geqslant 0$ and $\alpha + \beta \leqslant 3/8$ such that

$$\begin{aligned} \mathsf{G}(\vec{u}) = & \frac{4(\alpha+\beta)-1}{16} \left[\vec{u}\cdot\vec{\sigma} \otimes \vec{u}\cdot\vec{\sigma} - (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) \right] \\ & + \frac{\alpha-\beta}{4\sqrt{3}} \left(\vec{u}\cdot\vec{\sigma} \otimes \mathrm{id} + \mathrm{id} \otimes \vec{u}\cdot\vec{\sigma} \right) + \frac{1}{8} \mathrm{id} \otimes \mathrm{id} \end{aligned} \tag{3}$$

for all $\vec{u} \in \Omega$.

The margins

The margin of the above O-covariant observable is

$$\mathsf{G}^{[1]}(x) = \sum_{y,z\in\{+1,-1\}} \mathsf{G}(x,y,z) = \frac{(\alpha-\beta)x}{\sqrt{3}}(\sigma_x \otimes \mathrm{id} + \mathrm{id} \otimes \sigma_x) + \frac{1}{2}\,\mathrm{id} \otimes \mathrm{id}\,.$$

Comparing this formula with

$$\widetilde{\mathsf{X}}_a(\pm 1) = \mathrm{id} \odot \mathsf{X}_a(\pm 1) = \frac{1}{2}(\mathrm{id} \otimes \mathsf{X}_a(\pm 1) + \mathsf{X}_a(\pm 1) \otimes \mathrm{id})$$
$$= \frac{1}{4}(2\,\mathrm{id} \otimes \mathrm{id} \pm a(\mathrm{id} \otimes \sigma_x + \sigma_x \otimes \mathrm{id}))$$

yields

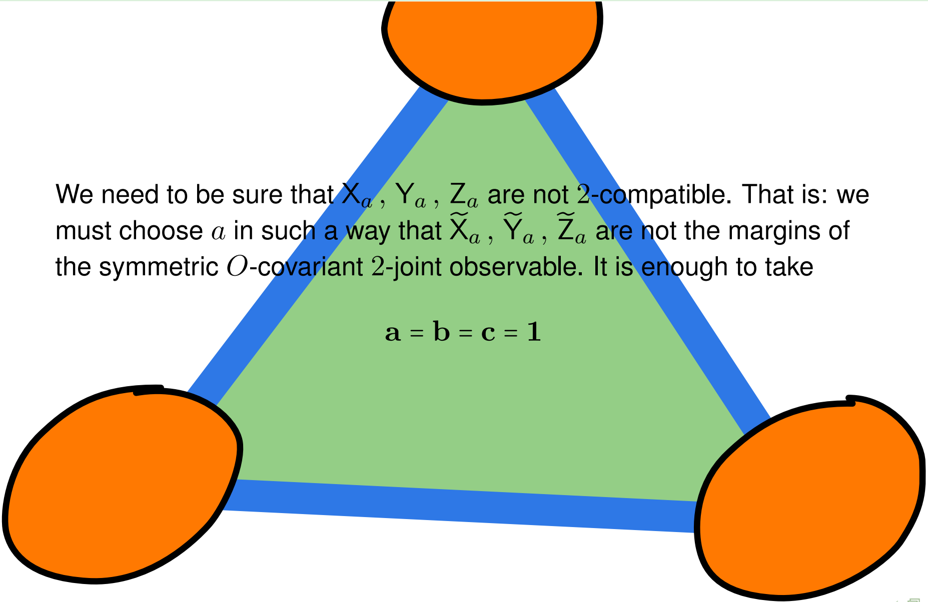
$$\alpha - \beta = \frac{\sqrt{3}}{4}\,a\,.$$

By the positivity conditions $\alpha \geqslant 0$, $\beta \geqslant 0$ and $\alpha + \beta \leqslant 3/8$, we thus see that the maximal value of a is $a = \sqrt{3}/2$.

Proposition

X_a, Y_a and Z_a are 2-compatible if and only if $0 \leqslant a \leqslant \sqrt{3}/2$.

Last compatibility stack in dimension 2



We need to be sure that X_a , Y_a , Z_a are not 2-compatible. That is: we must choose a in such a way that $\widetilde{\mathsf{X}}_a$, $\widetilde{\mathsf{Y}}_a$, $\widetilde{\mathsf{Z}}_a$ are not the margins of the symmetric O-covariant 2-joint observable. It is enough to take

$$\mathbf{a} = \mathbf{b} = \mathbf{c} = \mathbf{1}$$

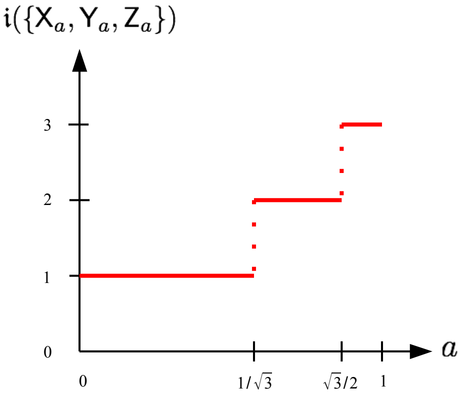


Figure: The index of incompatibility $\mathfrak{i}(\{X_a, Y_a, Z_a\})$ as a function of the noise parameter a for three noisy orthogonal qubit observables.

Thanks

References

- Kunjwal, R.; Heunen, C.; Fritz, T. Quantum realization of arbitrary joint measurability structures. *Phys. Rev. A* **2014**, *89*, 052126.
- Carmeli C.; Heinosaari T.; Reitzner D.; Schultz J.; Toigo A. Quantum Incompatibility in Collective Measurements. *Mathematics* **2016**, *4*, 052126.