How well can we approximate incompatible qubit observables (for a given measure)?

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Motivation

Incompatibility is an unavoidable feature of quantum theory.

We can circumvent this to some degree with compatible ‘approximation’ observables.

Quantifying how good an approximation we have requires an *error measure*.

By considering the qubit case, we will highlight the ‘optimal’ approximating observables found via two particular error measures for two incompatible sharp observables.

Preliminaries

Qubit systems ($\mathcal{H} = \mathbb{C}^2$)

States: $$\rho = \frac{1}{2}(I + \boldsymbol{r}\cdot\boldsymbol{\sigma}), \|\boldsymbol{r}\| \leq 1,$$

Effects: $$A = \frac{1}{2}(\alpha I + \boldsymbol{a}\cdot\boldsymbol{\sigma}),\ \|\boldsymbol{a}\| \leq \alpha \leq 2 - \|\boldsymbol{a}\|,$$

Binary Observables: $$\mathsf{C} : \pm \mapsto C_\pm = \frac{1}{2}((1 \pm \underbrace{\gamma}_{\textit{bias}}) \pm \underbrace{\boldsymbol{c}\cdot\boldsymbol{\sigma}}_{\textit{sharp}\text{ if } \|\boldsymbol{c}\| = 1}), \quad |\gamma| + \|\boldsymbol{c}\| \leq 1,$$

Unsharpness of C: $$U(\mathsf{C})^2 := 1 - \|\boldsymbol{c}\|^2.$$

Joint Measurability

Two binary observables C, D are jointly measurable if there exists

$$\mathsf{J} : \pm \times \pm \mapsto J_{\pm,\pm}$$

$$C_{\pm} = J_{\pm,+} + J_{\pm,-} \qquad\qquad D_{\pm} = J_{+,\pm} + J_{-,\pm}$$

Necessary (and sufficient for unbiased observables) condition:

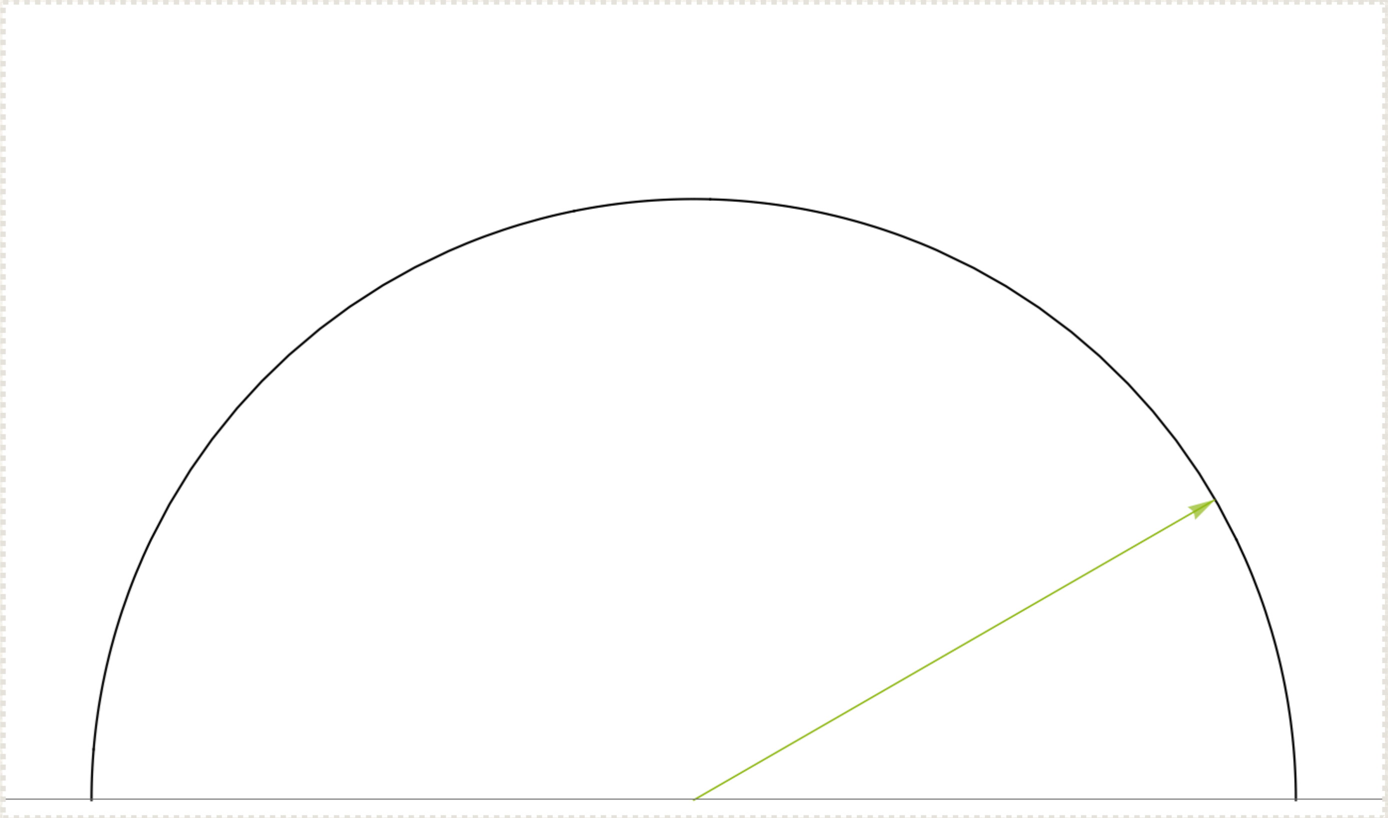
$$\|\boldsymbol{c} + \boldsymbol{d}\| + \|\boldsymbol{c} - \boldsymbol{d}\| \leq 2$$

Joint Measurability

$$\|\boldsymbol{c}+\boldsymbol{d}\| + \|\boldsymbol{c}-\boldsymbol{d}\| \leq 2 \quad \Leftrightarrow \quad \|\boldsymbol{c}\|^2 + \|\boldsymbol{d}\|^2 \leq 1 + (\boldsymbol{c}\cdot\boldsymbol{d})^2$$

Moving the vector changes the eccentricity of the ellipse.

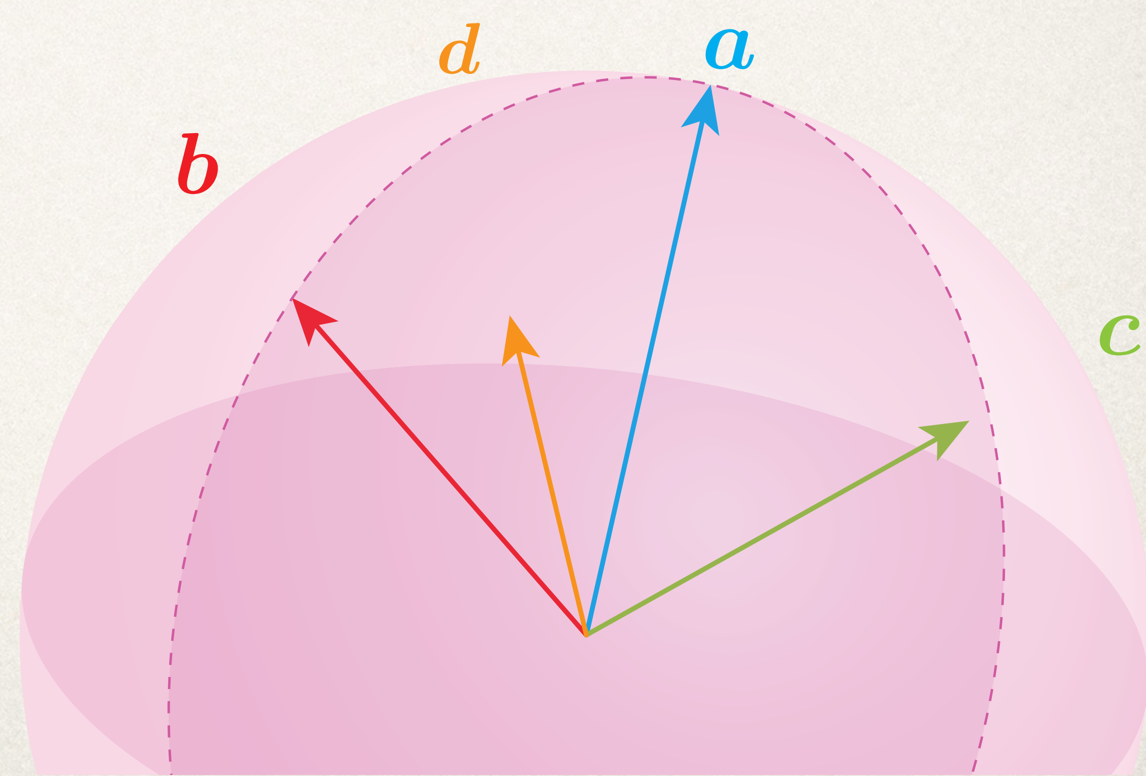
If the vector is normalised then the ellipse is just a line.



Joint Approximation and Error

$$\begin{tikzcd}
 & \mathsf{C} \arrow[r, squiggly, no head] & \mathsf{A} & \\
\mathsf{J} \arrow[ur] \arrow[dr] & & & \text{Approximation error} \arrow[ul] \arrow[dl] \\
 & \mathsf{D} \arrow[r, squiggly, no head] & \mathsf{B} &
\end{tikzcd}$$

Given some error measure, what is the best choice of C and D for approximating A and B?



How Do We Quantify Error?

There exist two main philosophies for quantifying error:

1. Value Comparison

2. Distribution Comparison

Value Comparison

One may wish to quantify error by how close the measurement outcomes of an approximating observable follow a target.

Measuring the target and then the approximating observable gives pairs of values whose deviation could serve as a measure of error.

This is commonly given by

$$\mathrm{tr}[\rho(\mathsf{A}[1] - \mathsf{C}[1])^2] =: \mathrm{tr}[\rho N_A^2]$$

First moment operators

Noise operator

Value Comparison

This method, however, is reliant on the target and approximating observables being jointly measurable, which is a significant restriction.

In the qubit case, this would force approximating observables to have Bloch vectors parallel/antiparallel to the target observables.

Distribution Comparison

This method takes the measurement statistics of two observables and sees how these distributions vary.

Since no direct comparison of measurement outcomes is made, the observables can be measured independently.

This removes the requirement that approximating observables are compatible with their targets.

How Do We Quantify Error?

There exist two main philosophies for quantifying error:

1. ~~Value Comparison~~

2. Distribution Comparison

Error Measures

We will consider two examples of distribution comparison errors:

1. Metric error $\mathcal{D}$

2. Calibration error Δ_{cal}

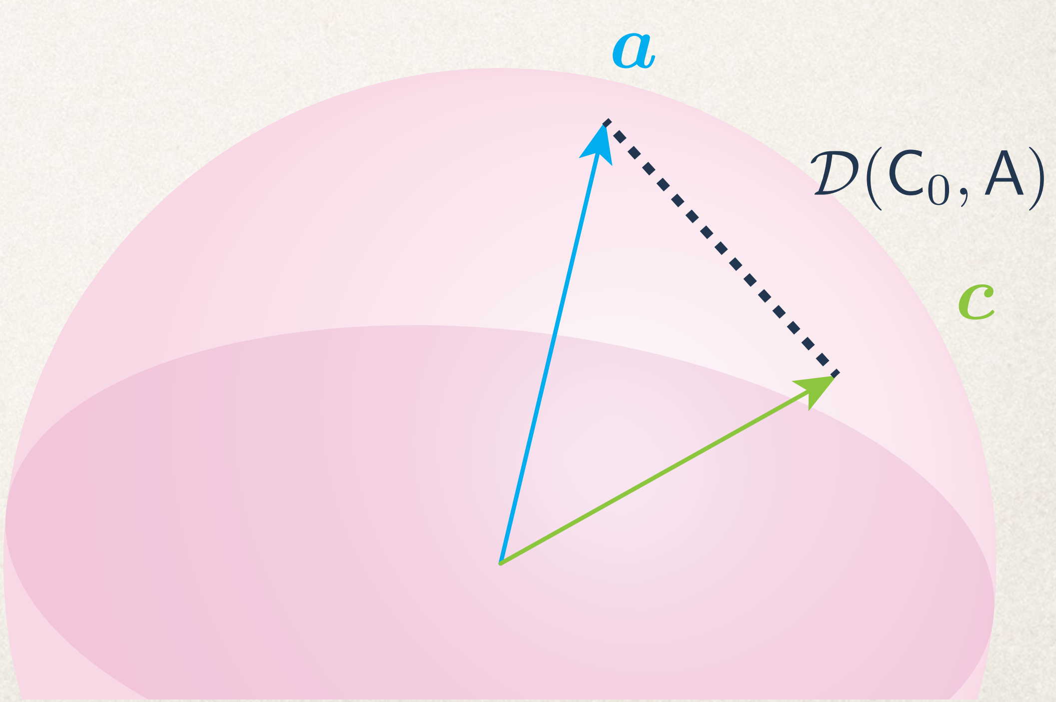
Metric Error

$$\mathcal{D}(\mathsf{C},\mathsf{A}) = 2\max_j \sup_\rho |\mathrm{tr}[\rho(A_j - C_j)]|$$

Measures the maximum deviation in the statistics of the two observables.

If we just consider binary observables...

$$\begin{aligned}\mathcal{D}(\mathsf{C},\mathsf{A}) &= 2\|A_+ - C_+\| \\ &= |\gamma| + \|\boldsymbol{a} - \boldsymbol{c}\|\end{aligned}$$



Best case with unbiased approximation

Metric Error

The metric error is half the square of the error quantity Δ based on the Wasserstein 2-distance for binary qubit observables [1].

Both Δ and $\mathcal{D}$ take the supremum over all states, but we may restrict ourselves to just the eigenstates of A. This leads us to consider *calibration error.*

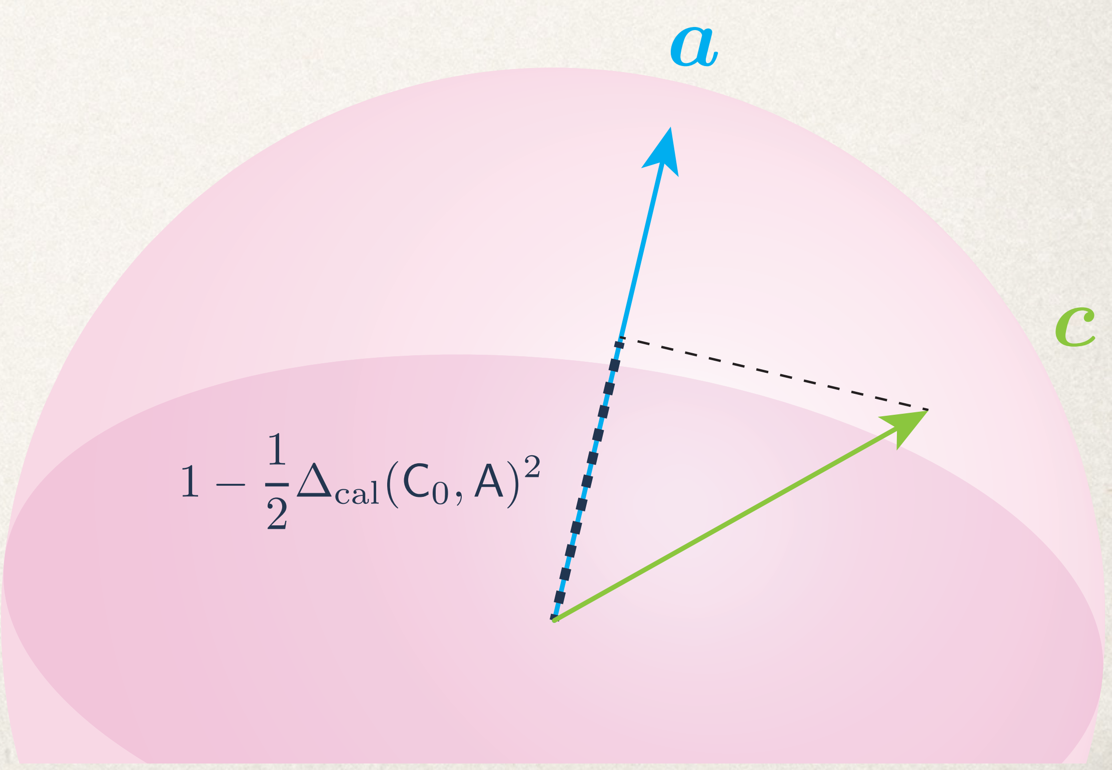
[1] P. Busch, P. Lahti and R. Werner, *Phys. Rev. A* 89, 012129 (2014).

Calibration Error

$$\Delta_{\mathrm{cal}}(\mathsf{C}, \mathsf{A})^2 = 2|\gamma| + 2(1 - \boldsymbol{a} \cdot \boldsymbol{c})$$

The error is again minimised when the approximating observable is unbiased.

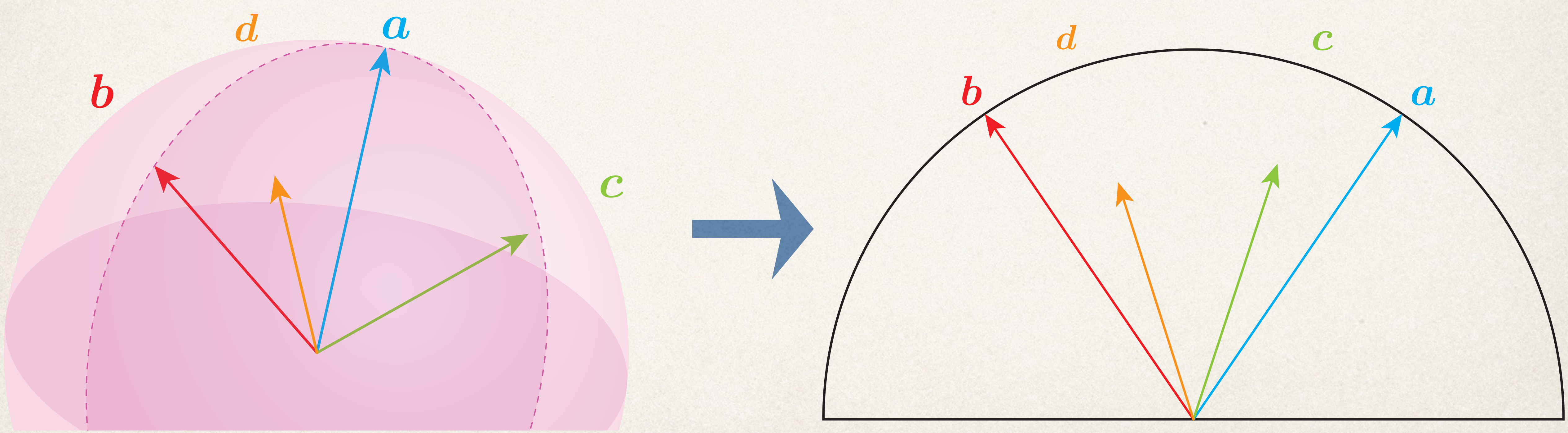
The unbiased case coincides with the noise measure quantity.



Time to optimise!

Optimisation of Metric Error

The first thing to note is that we can restrict ourselves to approximating observables in the plane spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$ [1].



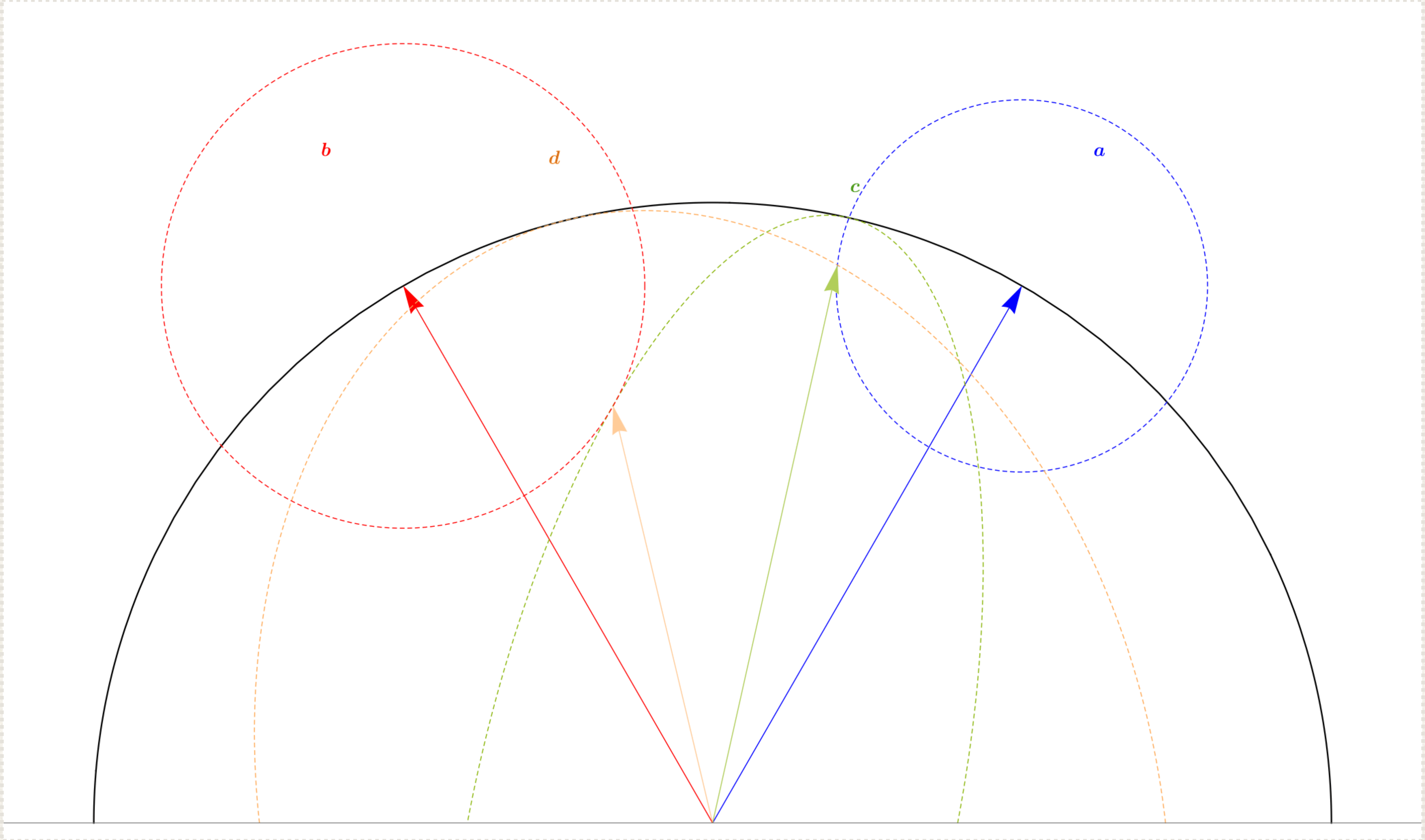
[1] P. Busch and T. Heinosaari, *Quant. Inf. Comp.* 8, 0797 (2008).

Optimisation of Metric Error

The joint measurability criterion restricts the approximators: making one sharper will reduce the sharpness of the other, and force the two vectors closer.

However, the metric error defines a circle of fixed value around the target observables. This means that for a given value $\mathcal{D}(\mathsf{C}_0, \mathsf{A})$ we are free to choose $\boldsymbol{c}$ that allows for the smallest $\mathcal{D}(\mathsf{D}_0, \mathsf{B})$.

Optimisation of Metric Error



Optimisation of Metric Error

The Lagrange multiplier method is used to minimise the metric error under the constraint of joint measurability [1].

The optimal approximators occur where the fixed error circles are tangent to the joint measurability ellipses.

[1] S. Yu, and C. H. Oh, *arXiv*: 1402.3785 (2014).

Optimisation of Metric Error

The optimal approximators are expressed parametrically [1]:

$$\boldsymbol{c} = \frac{(\mathcal{D}(\mathsf{D}_0, \mathsf{B}) + (1 - (\boldsymbol{c}\cdot\boldsymbol{d})^2)\cos\varphi)\sin\varphi\ \boldsymbol{a} + (\boldsymbol{c}\cdot\boldsymbol{d})\mathcal{D}(\mathsf{C}_0, \mathsf{A})\cos\varphi\ \boldsymbol{b}}{\sin\theta}$$

$$\boldsymbol{d} = \frac{(\mathcal{D}(\mathsf{C}_0, \mathsf{A}) + (1 - (\boldsymbol{c}\cdot\boldsymbol{d})^2)\sin\varphi)\cos\varphi\ \boldsymbol{b} + (\boldsymbol{c}\cdot\boldsymbol{d})\mathcal{D}(\mathsf{D}_0, \mathsf{B})\sin\varphi\ \boldsymbol{a}}{\sin\theta}$$

$$\sin\theta = \|\boldsymbol{a}\times\boldsymbol{b}\|, \quad \sin\varphi = \sqrt{\frac{1 - \|\boldsymbol{d}\|^2}{1 - (\boldsymbol{c}\cdot\boldsymbol{d})^2}}, \quad \cos\varphi = \sqrt{\frac{1 - \|\boldsymbol{c}\|^2}{1 - (\boldsymbol{c}\cdot\boldsymbol{d})^2}}$$

[1] S. Yu, and C. H. Oh, *arXiv*: 1402.3785 (2014).

Optimisation of Metric Error

Since the optimal approximators lie on the surface of their respective joint measurability ellipses, we know that

$$\|\boldsymbol{c}\|^2 + \|\boldsymbol{d}\|^2 = 1 + (\boldsymbol{c} \cdot \boldsymbol{d})^2$$

This then allows us to express $\sin\varphi$ and $\cos\varphi$ in terms of the unsharpness of C_0 and D_0:

$$\sin\varphi = \frac{U(\mathsf{C}_0)}{\sqrt{U(\mathsf{C}_0)^2 + U(\mathsf{D}_0)^2}}, \quad \cos\varphi = \frac{U(\mathsf{D}_0)}{\sqrt{U(\mathsf{C}_0)^2 + U(\mathsf{D}_0)^2}}$$

Optimisation of Calibration Error

Can we perform a similar analysis for calibration error?

Not exactly.

Optimisation of Calibration Error

If we use the Lagrange multiplier method to minimise the calibration error with the joint measurability constraint we find

$$\boldsymbol{a} = \frac{\boldsymbol{c} - (\boldsymbol{c} \cdot \boldsymbol{d})\boldsymbol{d}}{(1 - (\boldsymbol{c} \cdot \boldsymbol{d})^2)\sin\varphi}, \quad \boldsymbol{b} = \frac{\boldsymbol{d} - (\boldsymbol{c} \cdot \boldsymbol{d})\boldsymbol{c}}{(1 - (\boldsymbol{c} \cdot \boldsymbol{d})^2)\cos\varphi}$$

These vectors are orthogonal, so a local minima in general cannot be constructed.

Optimisation of Calibration Error

In the orthogonal case, the optimal approximating observables are simple to derive:

$$\boldsymbol{c} = \sin\varphi \ \boldsymbol{a} + (\boldsymbol{c}\cdot\boldsymbol{d})\cos\varphi \ \boldsymbol{b}$$
$$\boldsymbol{d} = \cos\varphi \ \boldsymbol{b} + (\boldsymbol{c}\cdot\boldsymbol{d})\sin\varphi \ \boldsymbol{a}$$

Optimisation of Calibration Error

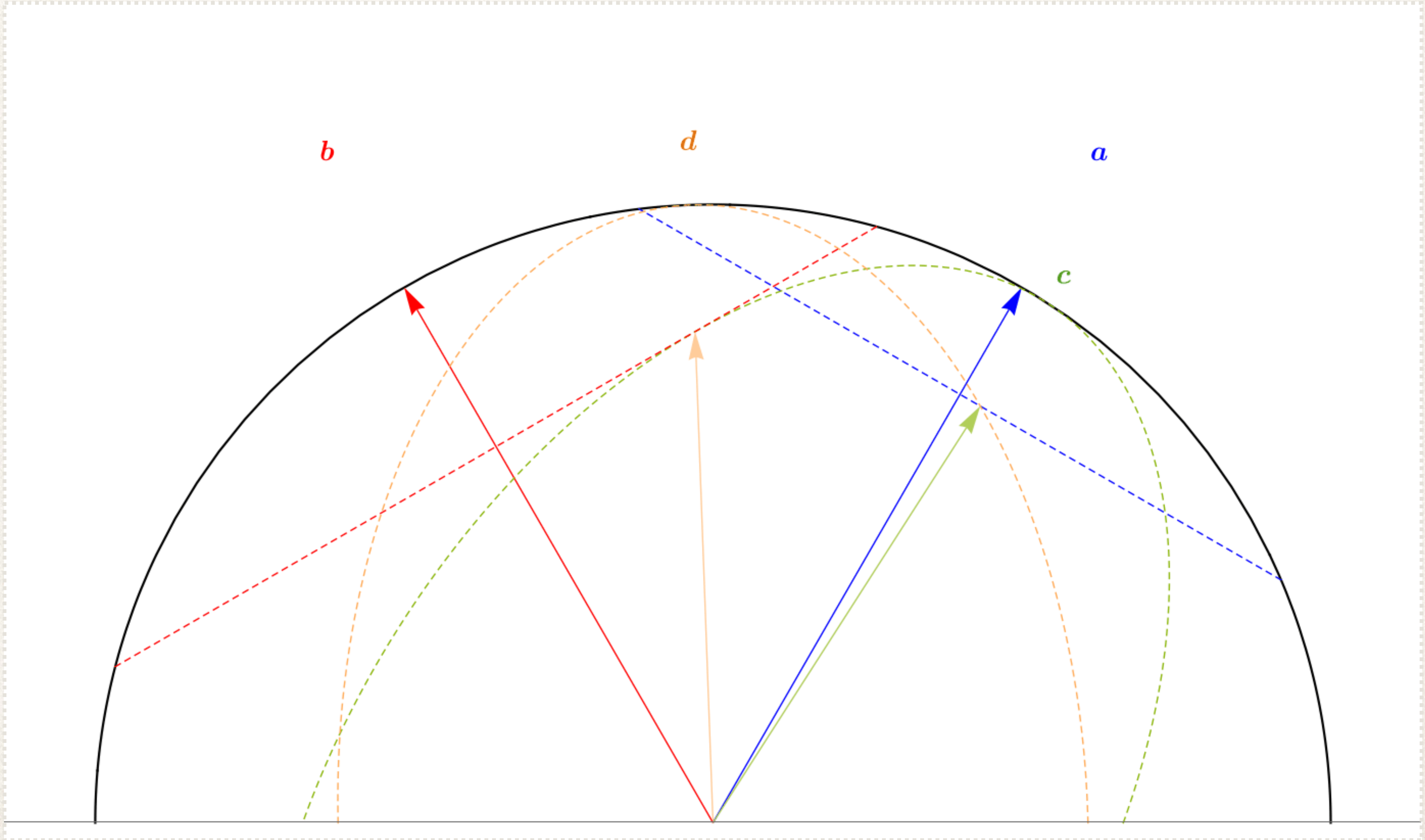
So what is happening for the calibration error?

While the metric error defines a circle of fixed error, the calibration error defines a fixed *line*.

In this case we now need the joint measurability ellipses to be tangent to these lines of fixed error.

Optimisation of Calibration Error

Optimising Δ_{cal} pushes $\boldsymbol{c}$ and $\boldsymbol{d}$ to the surface of the Bloch sphere.



Thank you for listening!

T. Bullock and P. Busch, *arXiv:* 1512.00104v2 (soon to be v3!)
http://demonstrations.wolfram.com/OptimalJointMeasurementsOfQubitObservables/