Quantum Measurement Uncertainty

New Relations for Qubits

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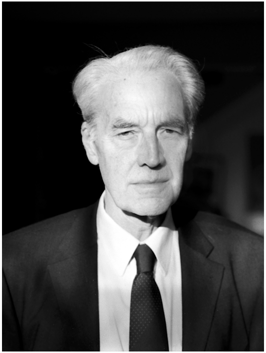

Peter Mittelstaedt 1929-2014

Outline

Heisenberg 1927

Essence of the quantum mechanical world view:

quantum uncertainty & *Heisenberg effect*

Heisenberg 1927

quantum uncertainty:

Preparation Uncertainty Relation: PUR

For any wave function ψ:

$$(\text{Width of } Q \text{ distribution}) \cdot (\text{Width of } P \text{ distribution}) \sim \hbar$$

(Heisenberg just discusses a Gaussian wave packet.)

Later generalisation:

$$\Delta_\rho A \, \Delta_\rho B \geq \tfrac{1}{2} \langle [A, B] \rangle_\rho$$

(Heisenberg didn't state this...)

Heisenberg 1927

Heisenberg effect:

- any measurement disturbs the object: *uncontrollable state change*
- measurements disturb each other: *quantum incompatibility*

Measurement Uncertainty Relation: MUR

$$(\textsc{Error of } Q \textsc{ measurement}) \cdot (\textsc{Error of } P) \sim \hbar$$

$$(\textsc{Error of } Q \textsc{ measurement}) \cdot (\textsc{Disturbance of } P) \sim \hbar$$

Reading Heisenberg's thoughts?

Heisenberg allegedly claimed (and proved):

$$\varepsilon(A,\rho)\,\varepsilon(B,\rho) \;\geq\; \tfrac{1}{2}\big|\langle[A,B]\rangle_\rho\big| \quad (???)$$

MUR made precise?

Heisenberg's thoughts – or Heisenberg's spirit?

(combined joint measurement errors for A, B) $\geq$ (incompatibility of A, B)

True of false? **Needed:**

- precise notions of approximate measurement
- measure of approximation error
- measure of disturbance

Quantum uncertainty challenged

nature physics LETTERS
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Experimental demonstration of a universally valid error–disturbance uncertainty relation in spin measurements

Jacqueline Erhart[1], Stephan Sponar[1], Georg Sulyok[1], Gerald Badurek[1], Masanao Ozawa[2] and Yuji Hasegawa[1]*

The uncertainty principle generally prohibits simultaneous measurements of certain pairs of observables and forms the basis of indeterminacy in quantum mechanics[1]. Heisenberg's original formulation, illustrated by the famous γ-ray microscope, sets a lower bound for the product of the measurement error and the disturbance[2]. Later, the uncertainty relation was reformulated in terms of standard deviations[3–5], where the focus was exclusively on the indeterminacy of predictions, whereas the unavoidable recoil in measuring devices has been ignored[6]. A correct formulation of the error–disturbance uncertainty relation, taking recoil into account, is essential for a deeper understanding of the uncertainty principle, as Heisenberg's original relation is valid only under specific circumstances[7–10]. A new error–disturbance relation, derived using the theory of general quantum measurements, has been claimed to be universally valid[11–14]. Here, we report a neutron-optical experiment that records the error of a spin-component measurement as well as the disturbance caused on another spin-component. The results confirm that both error and disturbance obey the new relation but violate the old one in a wide range of an experimental parameter.

The uncertainty relation was first proposed by Heisenberg[2] in 1927 as a limitation of simultaneous measurements of canonically conjugate variables owing to the back-action of the measurement: the measurement of the position Q of the electron with the error $\epsilon(Q)$, or 'the mean error', induces the disturbance $\eta(P)$, or 'the discontinuous change', of the momentum P so that they always satisfy the relation

$$\epsilon(Q)\eta(P) \sim \frac{h}{2} \qquad (1)$$

as $\sigma(A)^2 = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$. Note that a positive definite covariance term can be added to the right-hand side of equation (2), if squared, as discussed by Schrödinger[5]. For our experimental setting, this term vanishes. Robertson's relation (equation (2)) for standard deviations has been confirmed by many different experiments. In a single-slit diffraction experiment[15] the uncertainty relation, as expressed in equation (2), has been confirmed. A trade-off relation appears in squeezing coherent states of radiation fields[16], and many experimental demonstrations have been carried out[17].

Robertson's relation (equation (2)) has a mathematical basis, but has no immediate implications for limitations on measurements. This relation is naturally understood as limitations on state preparation or limitations on prediction from the past. On the other hand, the proof of the reciprocal relation for the error $\epsilon(A)$ of an A measurement and the disturbance $\eta(B)$ on observable B caused by the measurement, in a general form of Heisenberg's error–disturbance relation

$$\epsilon(A)\eta(B) \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \qquad (3)$$

is not straightforward, as Heisenberg's proof[2] used an unsupported assumption on the state just after the measurement[12], despite successful justifications for the Heisenberg-type relation for unbiased joint measurements[8–10]. Recently, rigorous and general theoretical treatments of quantum measurements have revealed the failure of Heisenberg's relation (equation (1)), and derived a new universally valid relation[11–14] given by

$$\epsilon(A)\eta(B) + \epsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \qquad (4)$$

Quantum uncertainty challenged

PRL **109**, 100404 (2012) PHYSICAL REVIEW LETTERS week ending 7 SEPTEMBER 2012

Violation of Heisenberg's Measurement-Disturbance Relationship by Weak Measurements

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While there is a rigorously proven relationship about uncertainties intrinsic to any quantum system, often referred to as "Heisenberg's uncertainty principle," Heisenberg originally formulated his ideas in terms of a relationship between the precision of a *measurement* and the disturbance it must create. Although this latter relationship is not rigorously proven, it is commonly believed (and taught) as an aspect of the broader uncertainty principle. Here, we experimentally observe a violation of Heisenberg's "measurement-disturbance relationship", using weak measurements to characterize a quantum system before and after it interacts with a measurement apparatus. Our experiment implements a 2010 proposal of Lund and Wiseman to confirm a revised measurement-disturbance relationship derived by Ozawa in 2003. Its results have broad implications for the foundations of quantum mechanics and for practical issues in quantum measurement.

Quantum uncertainty challenged

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Experimental violation and reformulation of the Heisenberg's error-disturbance uncertainty relation

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The uncertainty principle formulated by Heisenberg in 1927 describes a trade-off between the error of a measurement of one observable and the disturbance caused on another complementary observable such that their product should be no less than the limit set by Planck's constant. However, Ozawa in 1988 showed a model of position measurement that breaks Heisenberg's relation and in 2003 revealed an alternative relation for error and disturbance to be proven universally valid. Here, we report an experimental test of Ozawa's relation for a single-photon polarization qubit, exploiting a more general class of quantum measurements than the class of projective measurements. The test is carried out by linear optical devices and realizes an indirect measurement model that breaks Heisenberg's relation throughout the range of our experimental parameter and yet validates Ozawa's relation.

Correspondence and requests for materials

Recent media hype: the end of quantum uncertainty?

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7 September 2012 Last updated at 17:24

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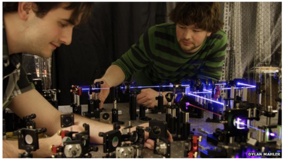
Heisenberg uncertainty principle stressed in new test

By Jason Palmer
Science and technology reporter, BBC News



The experiment requires preparing pairs of "entangled" photons, the particles from which light is made

Pioneering experiments have cast doubt on a founding idea of the branch of physics called quantum mechanics.

Related Stories

Quantenphysik

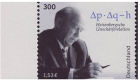
Der große Heisenberg irrte

17.11.2012 · Werner Heisenberg wollte seine berühmte Unbestimmtheitsbeziehung auch in den Störungen wiedererkennen, die ein Messung verursacht. Diesen Schluss haben kanadische Forscher widerlegt.

Von RAINER SCHARF

Artikel | Bilder (3) | Lesermeinungen (31)

Die von Werner Heisenberg 1927 formulierte Unschärfebeziehung ist trotz ihrer Tiefgründigkeit und Abstraktheit das wohl bekannteste Gesetz der Quantenphysik. Sie besagt vereinfacht, dass man nicht gleichzeitig die Geschwindigkeit und den Ort etwa eines Elektrons mit beliebiger Präzision bestimmen kann. Für die Popularität dieses Gesetzes hat vor allem eine ebenfalls von Heisenberg stammende bildhafte Erläuterung gesorgt,



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Werner Heisenberg und seine Unschärferelation sind sogar auf einer Briefmarke verewigt

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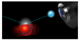
Synopsis: Rescuing Heisenberg

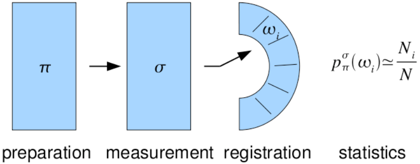


APS/Alan Stonebraker

Proof of Heisenberg's Error-Disturbance Relation
Paul Busch, Pekka Lahti, and Reinhard F. Werner
Phys. Rev. Lett. **111**, 160405 (2013)
Published October 17, 2013

The popular conception of the Heisenberg uncertainty principle is that measurement is unavoidably invasive. We disturb an object when we observe it, thus introducing error into subsequent measurements. However, recent experiments (see 6 September 2012 Synopsis) claim to have measurement errors below the Heisenberg limit. To address this apparent contradiction, a paper in *Physical Review Letters* reports a new formulation of the uncertainty principle in which measurement disturbance depends on the performance of the measuring device, which is quantified as the maximum possible change in the state of the object.

Quantum Measurement Statistics – Observables as POVMs



$$[\pi] \sim \rho, \quad [\sigma] \sim \mathsf{E} = \{\omega_i \mapsto E_i\} : \qquad p_\pi^\sigma(\omega_i) = \mathrm{tr}[\rho E_i] = p_\rho^{\mathsf{E}}(\omega_i)$$

$$\mathrm{POVM} : \qquad \mathsf{E} = \{E_1, E_2, \cdots, E_n\}, \quad 0 \le O \le E_i \le I\,, \quad \sum E_i = I$$

state changes: *instrument* $\omega_i, \rho \to \mathcal{I}_i(\rho)$

measurement processes: measurement scheme $\mathcal{M} = \langle \mathcal{H}_a, \phi, U, Z_a \rangle$

Signature of an observable: its statistics

$$p_\rho^{\mathsf{C}} \;=\; p_\rho^{\mathsf{A}} \quad \text{for all } \rho \quad \Longleftrightarrow \quad \mathsf{C} = \mathsf{A}$$

Minimal indicator for a measurement of C *to be a good approximate measurement of* A*:*

$$p_\rho^{\mathsf{C}} \;\simeq\; p_\rho^{\mathsf{A}} \quad \text{for all } \rho$$

Unbiased approximation – absence of systematic error:

$$\mathsf{C}[1] = \sum_j c_j C_j = \mathsf{A}[1] = \sum_i a_i A_i = A$$

... often taken as sole criterion for a good measurement

Joint Measurability/Compatibility

Definition: joint measurability (compatibility)

Observables $\mathsf{C} = \{C_+, C_-\}, \quad \mathsf{D} = \{D_+, D_-\}$ are *jointly measurable* if they are margins of an observable $\mathsf{G} = \{G_{++}, G_{+-}, G_{-+}, G_{--}\}$:

$$C_k = G_{k+} + G_{k-}, \quad D_\ell = G_{+\ell} + G_{-\ell}$$

Theorem

If one of C, D *is sharp (projection valued), then these observables are jointly measurable iff they commute:*

$$[C_k, D_\ell] = 0$$

Joint measurability in general

Pairs of **unsharp** observables may be jointly measurable
– **even when they do not commute!**

Approximate joint measurement: concept

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ C \arrow[d, squiggly] & & D \arrow[d, squiggly] \\ A & & B \end{tikzcd}$$

joint observable

approximator observables (compatible)

target observable

Task: find suitable measures of *approximation errors*

Approximation error

(vc) *value comparison*

(*e.g.* rms) deviation of outcomes of a joint measurement: accurate reference measurement *together* with measurement to be calibrated, on *same* system

(dc) *distribution comparison*

(*e.g.* rms) deviation between distributions of *separate* measurements: accurate reference measurement and measurement to be calibrated, applied to *separate* but identical ensembles

alternative measures of deviation: error bar width; relative entropy; *etc.* ...

Crucial:

Value comparison is of *limited applicability* in quantum mechanics!

Approximation error – Take 1: value comparison

Measurements/observables to be compared:

$$\mathsf{A} = \{a_1, a_2, \ldots, a_m\}, \quad \mathsf{C} = \{c_1, c_2, \ldots, c_n\}$$

where A is a sharp *(target)* observable
and C an *(approximator)* observable representing an approximate measurement of A

Protocol: *measure both* A *and* C *jointly on each system of an ensemble of identically prepared systems*

Proviso: This requires A and C to be compatible, hence commuting.

$$\delta_{\mathrm{vc}}(\mathsf{C}, \mathsf{A}; \rho)^2 \;=\; \sum_i (a_i - c_j)^2 \,\mathrm{tr}[\rho A_i C_j]$$

(Ozawa 1991)

Issue: $\delta_{\rm vc}$ is of limited use!
Attempted generalisation: *measurement noise* (Ozawa 2003)

$$\delta_{\rm vc}(\mathsf{C},\mathsf{A};\rho)^2 \;=\; \left\langle \mathsf{C}[2]-\mathsf{C}[1]^2\right\rangle_\rho + \left\langle (\mathsf{C}[1]-A)^2\right\rangle_\rho \;=\; \varepsilon_{\rm mn}(\mathsf{C},\mathsf{A};\rho)^2$$

where $\mathsf{C}[k]=\sum_j c_j^k C_j$, $A=\mathsf{A}[1]$ are the $k^{\rm th}$ moment operators...
...then give up assumption of commutativity of A, C

Critique (BLW 2013, 2014)

If A, C do not commute, then:

- $\delta_{\rm vc}(\mathsf{C},\mathsf{A};\rho)$ loses its meaning as rms value deviation
- and becomes unreliable as error indicator
  – *e.g.*, it is possible to have $\varepsilon_{\rm mn}(\mathsf{C},\mathsf{A};\rho)=0$ where A, C may not even have the same values.

Measurement noise as approximation error?

$$\varepsilon(C, A; \varphi)^2 = \langle \varphi \otimes \phi \,|\, (Z_\tau - A)^2 \varphi \otimes \phi \rangle \;\equiv \varepsilon_a^2$$

In general, pointer Z_τ and target observable A may not commute.

Compare to measuring the energy

$$H \;=\; \frac{P^2}{2m} + V(Q)$$

You can't measure H by measuring kinetic and potential energy and adding the outcomes.

Similarly: there's no justification for the assumption that $(Z_\tau - A)^2$ holds information about the mean squared differences between values of A, Z_τ.

Underlying quantum feature: *Heisenberg effect*.

Not Heisenberg's inequality: its true origin

Joint measurability and intrinsic noise/unsharpness

For compatible C, D:

$$\left(\langle \mathsf{C}[2]\rangle_\rho - \langle \mathsf{C}[1]^2\rangle_\rho\right)\left(\langle \mathsf{D}[2]\rangle_\rho - \langle \mathsf{D}[1]^2\rangle_\rho\right) \;\geq\; \tfrac{1}{4}\left|\left\langle [\mathsf{C}[1], \mathsf{D}[1]]\right\rangle_\rho\right|^2$$

Interpretation: for C, D to be jointly measurable, their degrees of unsharpness are bounded by their noncommutativity.

Let C, D be *unbiased approximators (ua)* of sharp observables A, B, that is: $\mathsf{C}[1] = A$, $\mathsf{D}[1] = B$. Then:

$$\varepsilon(\mathsf{C}, \mathsf{A}; \rho)^2\, \varepsilon(\mathsf{D}, \mathsf{B}; \rho)^2 \geq_{(\mathrm{ua})} \tfrac{1}{2}\left|\left\langle [A, B]\right\rangle_\rho\right|^2$$

Primarily a noise relation, not about error (and disturbance)

In particular: $\varepsilon(Q;\rho)\,\varepsilon(P;\rho) \not\geq \frac{\hbar}{2}$ – *unless* (ua) applies.

Ozawa and Branciard inequalities

$$\varepsilon(\mathsf{A},\rho)\,\varepsilon(B,\rho) + \varepsilon(A,\rho)\Delta_\rho B\ , + \Delta_\rho A\varepsilon(B,\rho) \ \geq\ \tfrac{1}{2}\big|\langle[A,B]\rangle_\rho\,,$$

$$\varepsilon(A)^2(\Delta_\rho B)^2 + \varepsilon(B)^2(\Delta_\rho A)^2 \\ + 2\sqrt{(\Delta_\rho A)^2(\Delta_\rho B)^2 - \tfrac{1}{4}|\langle[A,B]\rangle_\rho|^2}\,\varepsilon(A)\varepsilon(B) \ \geq\ \tfrac{1}{4}|\langle[A,B]\rangle_\rho|^2$$

Comments:

- Does allow for $\varepsilon(\mathsf{A};\rho)\,\varepsilon(\mathsf{B};\rho)\ <\ \frac{1}{2}|\langle[A,B]\rangle_\rho|$.
- Branciard's inequality is known to be *tight* for pure states.
- *Not unequivocally error tradeoff relations!* (BLW 2014)

Approximation error – Take 2: distribution comparison

Protocol: compare distributions of A and C as they are obtained in separate runs of measurements on two ensembles of systems in state ρ

$$\delta_\gamma(p^{\mathsf{C}}_\rho, p^{\mathsf{A}}_\rho)^\alpha \;=\; \textstyle\sum_{ij}(a_i - c_j)^\alpha \gamma(i,j) \quad (1 \le \alpha < \infty)$$

where γ is any joint distribution of the values of A and C with marginal distributions $p^{\mathsf{A}}_\rho,\;\; p^{\mathsf{C}}_\rho$

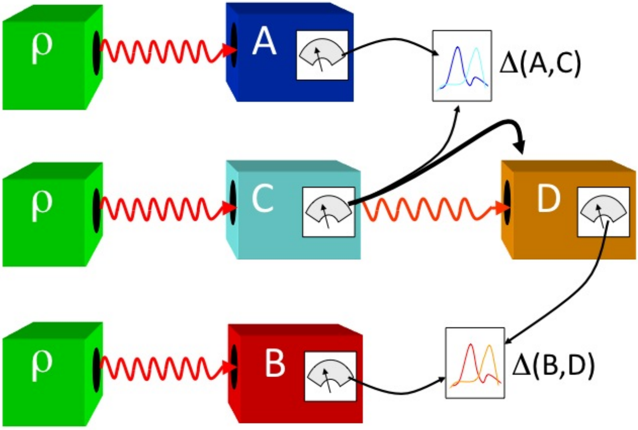
$$\Delta_\alpha(p^{\mathsf{C}}_\rho, p^{\mathsf{A}}_\rho) \;=\; \inf_\gamma \delta_\gamma(p^{\mathsf{C}}_\rho, p^{\mathsf{A}}_\rho)$$

Wasserstein-α distance – *scales with distances between points.*

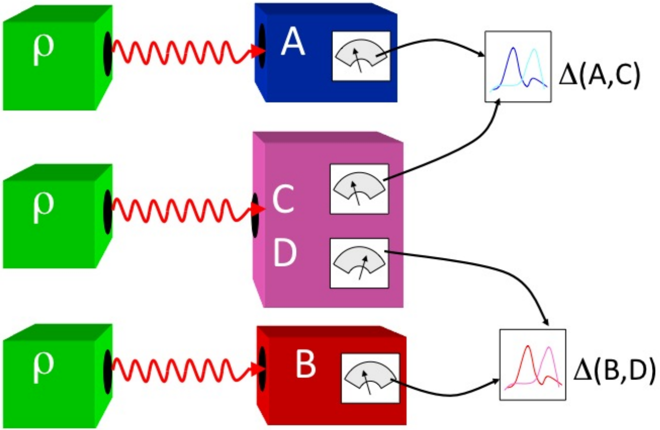
$$\Delta_\alpha(\mathsf{C}, \mathsf{A}) \;=\; \sup_\rho \Delta_\alpha(p^{\mathsf{C}}_\rho, p^{\mathsf{A}}_\rho)$$

quantum rms error: $\alpha = 2$

Disturbance



Disturbance quantified as approximation error



Qubits

$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ (Pauli matrices acting on $\mathbb{C}^2$)

- *States:* $\rho = \frac{1}{2}(I + \boldsymbol{r} \cdot \boldsymbol{\sigma}), \quad |\boldsymbol{r}| \leq 1$
- *Effects:* $A = \frac{1}{2}(a_0 I + \boldsymbol{a} \cdot \boldsymbol{\sigma}) \in [O, I], \quad 0 \leq \frac{1}{2}(a_0 \pm |\boldsymbol{a}|) \leq 1$
- *observables:* $(\Omega = \{+1, -1\})$

$$
\begin{aligned}
\mathrm{A}: \ \pm 1 &\mapsto A_\pm = \tfrac{1}{2}(I \pm \boldsymbol{a} \cdot \boldsymbol{\sigma}) \quad |\boldsymbol{a}| = 1 \\
\mathrm{B}: \ \pm 1 &\mapsto B_\pm = \tfrac{1}{2}(I \pm \boldsymbol{b} \cdot \boldsymbol{\sigma}) \quad |\boldsymbol{b}| = 1 \\
\mathrm{C}: \ \pm 1 &\mapsto C_\pm = \tfrac{1}{2}(1 \pm \gamma)\, I \pm \tfrac{1}{2}\boldsymbol{c} \cdot \boldsymbol{\sigma} \quad |\gamma| + |\boldsymbol{c}| \leq 1 \\
\mathrm{D}: \ \pm 1 &\mapsto D_\pm = \tfrac{1}{2}(1 \pm \delta)\, I \pm \tfrac{1}{2}\boldsymbol{d} \cdot \boldsymbol{\sigma} \quad |\delta| + |\boldsymbol{d}| \leq 1
\end{aligned}
$$

symmetric: $\gamma = 0$
sharp: $\gamma = 0$, $|\boldsymbol{c}| = 1$; $\quad \rightarrow \quad$ *unsharpness:* $U(\mathrm{C})^2 = 1 - |\boldsymbol{c}|^2$

Joint measurability of C, D

Symmetric case (sufficient for optimal compatible approximations):

Proposition

$\mathsf{C} = \{C_\pm = \frac{1}{2}(I \pm \boldsymbol{c}\cdot\boldsymbol{\sigma})\}$, $\mathsf{D} = \{D_\pm = \frac{1}{2}(I \pm \boldsymbol{d}\cdot\boldsymbol{\sigma})\}$ are compatible if and only if

$$|\boldsymbol{c}+\boldsymbol{d}| + |\boldsymbol{c}-\boldsymbol{d}| \le 2.$$

Interpretation: *unsharpness* $U(\mathsf{C})^2 = 1 - |\boldsymbol{c}|^2$; $|\boldsymbol{c}\times\boldsymbol{d}| = 2\big\|[C_+, D_+]\big\|$

$$|\boldsymbol{c}+\boldsymbol{d}| + |\boldsymbol{c}-\boldsymbol{d}| \le 2 \;\Leftrightarrow\; \left(1-|\boldsymbol{c}|^2\right)\left(1-|\boldsymbol{d}|^2\right) \ge |\boldsymbol{c}\times\boldsymbol{d}|^2$$

$$\mathsf{C},\mathsf{D} \text{ compatible } \;\Leftrightarrow\; U(\mathsf{C})^2 \times U(\mathsf{D})^2 \ge 4\big\|[C_+, D_+]\big\|^2$$

Approximation error

Recall: Observable C is a good approximation to A if $p_\rho^{\mathsf{C}} \simeq p_\rho^{\mathsf{A}}$

Take here: probabilistic distance

$$d_p(\mathsf{C},\mathsf{A}) = \sup_\rho \sup_X \left| \mathrm{tr}[\rho \mathsf{C}(X)] - \mathrm{tr}[\rho \mathsf{A}(X)] \right| = \sup_X \left\| \mathsf{C}(X) - \mathsf{A}(X) \right\|$$

Qubit case: $C_+ = \frac{1}{2}\big(c_0 I + \boldsymbol{c}\cdot\boldsymbol{\sigma}\big)$, $A_+ = \frac{1}{2}\big(a_0 I + \boldsymbol{a}\cdot\boldsymbol{\sigma}\big)$

$$d_p(\mathsf{C},\mathsf{A}) = \left\| C_+ - A_+ \right\| = \tfrac{1}{2}|c_0 - a_0| + \tfrac{1}{2}|\boldsymbol{c} - \boldsymbol{a}| \;\equiv\; d_a \;\in\; [0,1].$$

Comparison 1: Wasserstein 2-distance (quantum rms error)

$$\Delta_2\left(p_\rho^{\mathsf{C}}, p_\rho^{\mathsf{A}}\right)^2 = \inf_\gamma \sum_{ij} (a_i - c_j)^2 \gamma(i,j)$$

where γ runs through all joint distributions with margins p_ρ^{C}, p_ρ^{A}.

$$\Delta_2(\mathsf{C},\mathsf{A})^2 = \sup_\rho d_2\left(p_\rho^{\mathsf{C}}, p_\rho^{\mathsf{A}}\right)^2 \equiv \Delta_a^2$$

Qubit case:

$$\begin{aligned}\Delta_a^2 = \Delta_2(\mathsf{C},\mathsf{A})^2 &= 2|c_0 - a_0| + 2|\boldsymbol{c} - \boldsymbol{a}| \\ &= 4d_p(\mathsf{C},\mathsf{A}) = 4d_a.\end{aligned}$$

Comparison 2: Measurement noise (Ozawa *et al*)

$$\varepsilon(\mathsf{C},\mathsf{A};\varphi)^2 = \langle \varphi\otimes\phi \,|\, (Z_\tau - A)^2 \varphi\otimes\phi\rangle$$
$$= \left\langle \mathsf{C}[2]-\mathsf{C}[1]^2\right\rangle_\rho + \left\langle (\mathsf{C}[1]-A)^2\right\rangle_\rho \equiv \varepsilon_a^2$$

Qubit observables, symmetric case:

$$\varepsilon_a^2 = 1 - |\boldsymbol{c}|^2 + |\boldsymbol{a}-\boldsymbol{c}|^2 = U(\mathsf{C})^2 + 4d_a^2$$

$\varepsilon(\mathsf{A};\rho)$ double counts contribution from unsharpness.

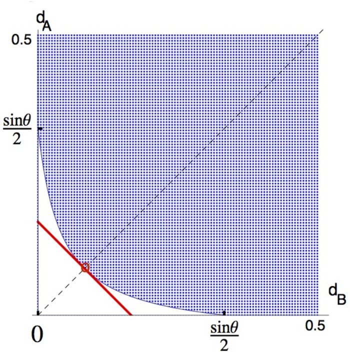
Optimising approximate joint measurements

$$
\begin{tikzcd}
 & G_{k\ell} \arrow[dl, "\sum_\ell"'] \arrow[dr, "\sum_k"] & \\
C_k \arrow[d, squiggly, "d_p(\mathsf{C},\mathsf{A})"'] & & D_\ell \arrow[d, squiggly, "d_p(\mathsf{D},\mathsf{B})"] \\
A_k & & B_\ell
\end{tikzcd}
$$

Goal

To make errors $d_{\mathsf{A}} = d_p(\mathsf{C}, \mathsf{A})$, $d_{\mathsf{B}} = d_p(\mathsf{D}, \mathsf{B})$ simultaneously as small as possible, subject to the constraint that C, D are compatible.

Admissible error region

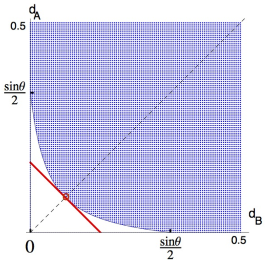


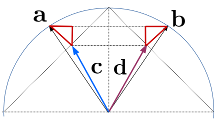
$\sin\theta = |\boldsymbol{a} \times \boldsymbol{b}|$

$(d_A, d_B) = (d_p(C, A), d_p(D, B)) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ with C, D compatible

trivial approximations: $C_+ = \gamma I$, $D_+ = \delta I$;
then $d_A = \max(\gamma, 1-\gamma) \geq \frac{1}{2}$, $d_B = \max(\delta, 1-\delta) \geq \frac{1}{2}$

Qubit Measurement Uncertainty Relation: Take 1





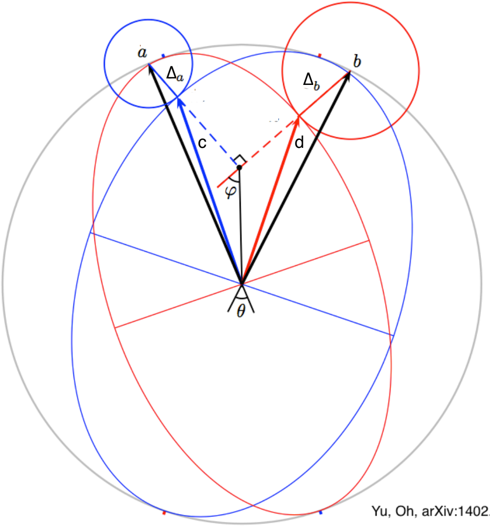
$$\sin\theta = |\boldsymbol{a} \times \boldsymbol{b}|$$

PB, T Heinosaari (2008), `arXiv:0706.1415`

$$\begin{aligned} |\boldsymbol{c}+\boldsymbol{d}| + |\boldsymbol{c}-\boldsymbol{d}| &\le 2 \\ U(\mathsf{C})^2 \times U(\mathsf{D})^2 &\ge 4\|[C_+, D_+]\|^2 \\ d_p(\mathsf{C},\mathsf{A}) + d_p(\mathsf{D},\mathsf{B}) &\ge \frac{1}{2\sqrt{2}}\left[\,|\boldsymbol{a}+\boldsymbol{b}| + |\boldsymbol{a}-\boldsymbol{b}| - 2\,\right] \end{aligned}$$

$$|\boldsymbol{a}+\boldsymbol{b}| + |\boldsymbol{a}-\boldsymbol{b}| = 2\sqrt{1+|\boldsymbol{a}\times\boldsymbol{b}|} = 2\sqrt{1+2\|[A_+, B_+]\|}$$

Qubit Measurement Uncertainty: Take 2 – boundary region



Yu, Oh, arXiv:1402.3785

Qubit Measurement Uncertainty

PB & T Heinosaari (2008), S Yu and CH Oh (2014)

Optimiser, case $\boldsymbol{a} \perp \boldsymbol{b}$:

$\boldsymbol{c} = |\boldsymbol{c}|\boldsymbol{a}, \quad \boldsymbol{d} = |\boldsymbol{d}|\boldsymbol{b},$

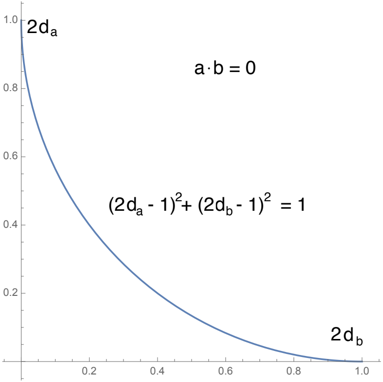
$2d_a = |\boldsymbol{a} - \boldsymbol{c}| = 1 - |\boldsymbol{c}|,$

$2d_b = |\boldsymbol{b} - \boldsymbol{d}| = 1 - |\boldsymbol{d}|,$

Compatibility constraint:

$|\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1, \quad \textit{i.e.}, \; U(\mathrm{C})^2 + U(\mathrm{D})^2 = 1$

$(1 - 2d_a)^2 + (1 - 2d_b)^2 = |\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1$



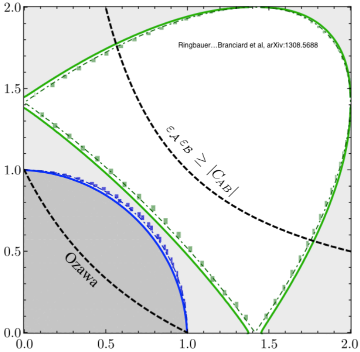
Ozawa–Branciard (C Branciard 2013, M Ringbauer *et al* 2014)

$$\varepsilon_a^2\left(1-\frac{\varepsilon_a^2}{4}\right) + \varepsilon_b^2\left(1-\frac{\varepsilon_b^2}{4}\right) \geq 1$$

$$\left(1-\frac{\varepsilon_a^2}{2}\right)^2 + \left(1-\frac{\varepsilon_b^2}{2}\right)^2 \leq 1$$

$$\varepsilon_a^2 \equiv 4d'_a, \quad \varepsilon_b^2 \equiv 4d'_b$$

$$(2d'_a-1)^2 + (2d'_b-1)^2 \leq 1$$



Optimiser: $\boldsymbol{c} = |\boldsymbol{c}|\boldsymbol{a}, \quad \boldsymbol{d} = |\boldsymbol{d}|\boldsymbol{b},$

Compatibility constraint: $|\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1$, *i.e.*, $U(\mathrm{C})^2 + U(\mathrm{D})^2 = 1$

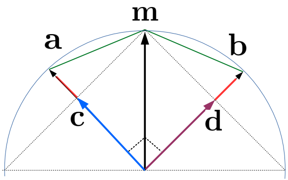
$4d'_a = \varepsilon_a^2 = 1 - |\boldsymbol{c}|^2 + |\boldsymbol{a}-\boldsymbol{c}|^2 = 2|\boldsymbol{a}-\boldsymbol{c}| = 4d_a, \quad 4d'_b = \varepsilon_b^2 = 4d_b$

$(2d_a-1)^2 + (2d_b-1)^2 = |\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1$

A twist: Ozawa's error

Branciard's inequality has another optimiser:
$\mathrm{M} = \{M_+, M_-\} = \mathrm{C}' = \mathrm{D}'$, $M\pm = \frac{1}{2}(I \pm \boldsymbol{m}\cdot\boldsymbol{\sigma})$

$\boldsymbol{m}$ "between" $\boldsymbol{a}, \boldsymbol{b}$



$$\varepsilon(\mathrm{M},\mathrm{A}) = \varepsilon(\mathrm{M},\mathrm{B}) = \varepsilon(\mathrm{A},\mathrm{C}) = \varepsilon(\mathrm{B},\mathrm{D})$$

but

$$2d_p(\mathrm{C},\mathrm{A}) = 2d_p(\mathrm{D},\mathrm{B}) = |\boldsymbol{a}-\boldsymbol{c}| \;<\; |\boldsymbol{a}-\boldsymbol{m}| = 2d_p(\mathrm{M},\mathrm{A}) = 2d_p(\mathrm{M},\mathrm{B})$$

Conclusion

(1) Heisenberg's spirit materialised

$$\big(\text{joint measurement errors for } \mathsf{A},\mathsf{B}\big) \;\geq\; \big(\text{incompatibility of } \mathsf{A},\mathsf{B}\big)$$

$$\big(\text{unsharpness of } \textit{compatible}\ \mathsf{C},\mathsf{D}\big) \;\geq\; \big(\text{noncommutativity of } \mathsf{C},\mathsf{D}\big)$$

Shown here for qubit observables.
Also known: case of position and momentum (BLW 2013):

$$\Delta_2(\mathsf{C},Q)\,\Delta_2(\mathsf{D},P) \;\geq\; \frac{\hbar}{2}$$

Generic results: finite dimensional Hilbert spaces, arbitrary discrete, finite-outcome observables (Miyadera 2011)

(2) Importance of judicious choice of error measure

- valid MURs obtained for Wasserstein-2 distance, error bar widths
- measurement noise / value comparison – **not suited** for universal MURs

References/Acknowledgements

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http://demonstrations.wolfram.com/HeisenbergTypeUncertaintyRelationForQubits/

$$\sim \quad \mathcal{Q}.\mathcal{E}.\mathcal{D}. \quad \sim$$