Problem 1

An electron is in the spin state

$$\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

- **Determine the normalization constant A.**

$$\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

Normalization of the state dictates that:

$$\chi^{+}\chi = 1$$

Hence,

$$|A|^2(9+16) = 1 \implies A = \frac{1}{5}$$

- **Find the expectation value of S_x, S_y and S_z.**

The expectation values of the components are given by:

$$< S_x >= \chi^{+} S_x \chi = \frac{1}{25}\,\frac{\hbar}{2} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$< S_x >= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 3i \end{pmatrix} = 0$$

$$< S_y >= \chi^{+} S_y \chi = \frac{1}{25}\,\frac{\hbar}{2} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$< S_y >= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} -4i \\ -3 \end{pmatrix} = \frac{-12\hbar}{25}$$

and

$$< S_z >= \chi^{+} S_z \chi = \frac{1}{25}\frac{\hbar}{2} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$< S_z >= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 3i \\ -4 \end{pmatrix} = \frac{-7\hbar}{50}$$

- **Find the "uncertainities" σ_{S_x}, σ_{S_y} and σ_{S_z}.(Note: These sigmas are standard deviations, not Pauli matrices!).**

$$\sigma_{S_x} = \sqrt{\sigma_{S_x}^2}$$

but, $\sigma_{S_x} =< S_x^2 > - < S_x >^2$, the same holds for σ_{S_y} and σ_{S_z}.
We know that $< S_x^2 >=< S_y^2 >=< S_z^2 >= \frac{\hbar^2}{4}$, this holds true for spin $\frac{1}{2}$ particles.

Using solution (b), one can see that:

$$\sigma_{S_x}^2 =< S_x^2 > - < S_x >^2 = \frac{\hbar^2}{4} - 0 \implies \boxed{\sigma_{S_x} = \frac{\hbar}{2}}$$

Similarly,

$$\sigma_{S_y}^2 =< S_y^2 > - < S_y >^2 = \frac{\hbar^2}{4} - \left[\frac{12}{25}\right]^2 \hbar^2 = \frac{49}{2500}\hbar^2 \implies \boxed{\sigma_{S_y} = \frac{7}{50}\hbar}$$

$$\sigma_{S_z}^2 =< S_z^2 > - < S_z >^2 = \frac{\hbar^2}{4} - \left[\frac{7}{50}\right]^2 \hbar^2 = \frac{\hbar^2}{2500}(625 - 49) = \frac{576}{2500}\hbar^2 \implies \boxed{\sigma_{S_z} = \frac{12}{25}\hbar}$$

- **Confirm that your results are consistent with all three uncertainity principles.**

$$\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}\frac{7}{50}\hbar \geq \frac{\hbar}{2}|S_z| = \frac{\hbar}{2}\frac{7}{50}\hbar \qquad \text{at the limit!}$$

$$\sigma_{S_y}\sigma_{S_x} = \frac{7}{50}\hbar\frac{12}{25}\hbar \geq \frac{\hbar}{2}|S_x| \qquad \text{clear!}$$

$$\sigma_{S_x}\sigma_{S_x} = \frac{12}{25}\hbar\frac{\hbar}{2} \geq \frac{\hbar}{2}|S_y| = \frac{\hbar}{2}\frac{12}{25}\hbar \qquad \text{at the limit!}$$

Problem 2

An electron is in the spin state

$$\chi = A\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$$

- **Determine the constant A by normalizing χ.**

  Normalizing the state yields:

$$\chi^+\chi = |A|^2(1+4+4) = 1 \implies A = \frac{1}{3}$$

hence, $\chi = \frac{1-2i}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a(\uparrow) + b(\downarrow)$

- **If you measured S_z on this electron, what values could you get, and what is the probability of each? What is the expectation value of S_z?**

$$\chi = \frac{1}{3}(1-2i)\left|\uparrow\right\rangle + \frac{2}{3}\left|\downarrow\right\rangle$$

If you measure S_z on this electron, you will get $\hbar/2$ with probability 5/9 and $-\hbar/2$ with probability of 4/9.
Hence,

$$< S_z >= \frac{5}{9}\Big(\frac{\hbar}{2}\Big) + \frac{4}{9}\Big(-\frac{\hbar}{2}\Big) = \frac{\hbar}{18}$$

- **If you measured S_x on this electron, what values could you get, and what is the probability of each? What is the expectation value of S_x?**

We know that :

$$< S_x >= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$< S_y >= \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

According to the generalized statistical interpretation, one can find the eigenvalues and eigenstates of S_x as:

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = 0 \implies \lambda^2 = \Big[\frac{\hbar}{2}\Big]^2 \implies \lambda = \pm\frac{\hbar}{2}$$

The eigenspinors are obtained using the following eigenvalue relation :

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \implies \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \pm\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \implies \beta = \pm\alpha$$

Hence, the normalized eigenspinors of S_x are:

$$\chi_+^{(x)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \text{ with eigenvalue} + \frac{\hbar}{2}$$

$$\chi_-^{(x)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \text{ with eigenvalue} - \frac{\hbar}{2}$$

The generic spinor χ now can be expressed as (see solution (a)).

$$\boxed{\chi = \Big(\frac{a+b}{\sqrt{2}}\Big)\chi_+^{(x)} + \Big(\frac{a-b}{\sqrt{2}}\Big)\chi_-^{(x)}}$$

If you measure S_x, the probability of getting $+\frac{\hbar}{2}$ is $\frac{1}{2}|a+b|^2 = \frac{13}{18}$.

Similarly the probability of getting $-\frac{\hbar}{2}$ is $\frac{1}{2}|a-b|^2 = \frac{5}{18}$.

Hence :

$$\boxed{< S_x >= \frac{13}{18}\frac{\hbar}{2} + \frac{5}{18}\Big(-\frac{\hbar}{2}\Big) = \frac{2\hbar}{9}}$$

- **If you measured S_y on this electron, what values could you get, and what is the probability of each? What is the expectation value of S_y?**

  Similarly using $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ one can find the probability of finding $\frac{\hbar}{2}$ is$\frac{17}{18}$ and $-\frac{\hbar}{2}$ with a probability of $\frac{1}{18}$.

  Hence :

$$\boxed{< S_y >= \frac{17}{18}\frac{\hbar}{2} + \frac{1}{18}\Big(-\frac{\hbar}{2}\Big) = \frac{4\hbar}{9}}$$

Problem 3

Construct the matrix S_r representing the component of spin angular momentum along an arbitrary direction $\hat{r}$. Use spherical coordinates, for which:

$$\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}.$$

Find the eigenvalues and (normalized) eigenspinors of S_r.

Consider an arbitrary direction defined by the unit vector $\hat{r}$.

$$\boxed{\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}}$$

The components of spin angular momentum along $\hat{r}$ direction $\hat{i}$.

$$\begin{aligned}
S_r = S.\hat{r} &= S_x \ \sin\theta\cos\phi + S_y \ \sin\theta\sin\phi + S_z \ \cos\theta \\
&= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 1 \\ 0 & -\cos\theta \end{pmatrix}\right] \\
&= \frac{\hbar}{2}\left[\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -cos\theta \end{pmatrix}\right] \\
&= \frac{\hbar}{2}\left[\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -cos\theta \end{pmatrix}\right]
\end{aligned}$$

$$\begin{vmatrix} (\frac{\hbar}{2}\cos\theta - \lambda) & \frac{\hbar}{2}e^{-i\phi}\sin\theta \\ \frac{\hbar}{2}e^{i\phi}\sin\theta & (-\frac{\hbar}{2}\cos\theta - \lambda) \end{vmatrix} = -\frac{\hbar^2}{4}\cos^2\theta + \lambda^2 - \frac{\hbar^2}{4}\sin^2\theta = 0 \implies \lambda^2 = \frac{\hbar^2}{4}\Big[\sin^2\theta + \cos^2\theta\Big] \implies \boxed{\lambda = \pm\frac{\hbar}{2}}$$

To find the eigenspinsors:

$$\frac{\hbar}{2}\begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

$$\alpha\cos\theta + \beta e^{-i\phi}\sin\theta = \pm\alpha$$

$$\beta = e^{i\phi}\,\frac{(\pm 1 - \cos\theta)}{\sin\theta}\,\alpha$$

Upper sign: Use $1-\cos\theta = 2\sin^2\theta/2$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, then $\beta = e^{i\phi}\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\,\alpha$
Normalization provides:

$$\begin{aligned}|\alpha|^2 + |\beta|^2 &= 1 \\ \implies |\alpha|^2 + \frac{\sin^2(\frac{\theta}{2})}{\cos^2(\frac{\theta}{2})}|\alpha|^2 &= 1 \\ \implies |\alpha|^2\frac{1}{\cos^2(\frac{\theta}{2})} &= 1 \implies \alpha = \cos\frac{\theta}{2}\end{aligned}$$

and

$$\beta = e^{i\phi}\sin\left(\frac{\theta}{2}\right)$$

Hence:

$$\boxed{\chi_+^{(r)} = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right)\end{pmatrix}}$$

Lower sign:
Use:

$$1+\cos\theta = 2\cos^2(\frac{\theta}{2})$$

$$\beta = -e^{i\phi}\frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\,\alpha$$

and

$$1 = |\alpha|^2 + \frac{\cos^2(\frac{\theta}{2})}{\sin^2(\frac{\theta}{2})}|\alpha|^2 = |\alpha|^2\frac{1}{\sin^2(\frac{\theta}{2})}$$

Use $\alpha = e^{-i\phi}\sin(\theta/2)$ and $\beta = -\cos(\theta/2)$ and:

$$\boxed{\chi_-^{(r)} = \begin{pmatrix}e^{-i\phi}\sin\left(\frac{\theta}{2}\right) \\ -\cos\left(\frac{\theta}{2}\right)\end{pmatrix}}$$

Problem 4

An electron is at rest in an oscillating magnetic field

$$B = B_0 \cos(\omega\ t)\hat{k},$$

Where B_0 and ω are constants.

- Construct the Hamiltonian matrix for this system.
- The electron starts out (at t = 0) in the spin-up state with respect to the x-axis (that is $\chi(0) = \chi_+^{(x)}$). Determine $\chi(t)$ at any subsequent time. Beware: This is a time-dependent Hamiltonian, so you cannot get $\chi(t)$ in the usual way from stationary states. Fortunately, in this case you can solve the time dependent Schrödinger equation directly.
- What is the minimum field (B_0) required to force a complete flip in S_x?

The Hamiltonian is given by :

$$H = -\gamma \vec{B}.S = -\gamma B_o \cos\omega\ t\ S_z = \frac{-\gamma B_0 \hbar}{2} \cos\omega\ t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi_{(t)} = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$$

and $\alpha(0) = \beta(0) = \frac{1}{\sqrt{2}}$.

$$i\hbar \frac{\partial \chi}{\partial t} = H\chi \implies i\hbar \frac{\partial \chi}{\partial t} = i\hbar \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix}$$

$$i\hbar \frac{\partial \chi}{\partial t} = i\hbar \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \frac{-\gamma B_0 \hbar}{2} \cos\omega\ t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{-\gamma B_0 \hbar}{2} \cos \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$$

$$\dot{\alpha} = i\Big(\frac{\gamma B_0}{2}\Big) \cos\omega\ t\ \alpha$$

$$\frac{d\alpha}{\alpha} = i(\frac{\gamma B_0}{2}) \cos\omega\ tdt$$

$$\ln(\alpha) = i\Big(\frac{\gamma B_0}{2}\Big) \frac{\sin\omega\ t}{\omega} + C$$

$$\alpha(t) = Ae^{i(\gamma B_0/2\omega)\sin\omega\ t}$$

$$\alpha(0) = A = \frac{1}{\sqrt{2}} \implies \alpha(t) = \frac{1}{\sqrt{2}}\ e^{i(\gamma B_0/2\omega)\sin\omega\ t}$$

$$\dot{\beta} = -i\Big(\frac{\gamma B_0}{2}\Big) \cos\omega\ t\ \beta \implies \beta(t) = \frac{1}{\sqrt{2}}\ e^{-i(\gamma B_0/2\omega)\sin\omega\ t}$$

$$\boxed{\chi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i(\gamma B_0/2\omega)\sin\omega\ t} \\ e^{-i(\gamma B_0/2\omega)\sin\omega\ t} \end{pmatrix} = a \uparrow + b \downarrow}$$

The probability of getting $\frac{-\hbar}{2} = \sin^2[\frac{\gamma B_0}{2\omega} \sin\omega\ t]$ (see solution 2).

The argument of $\sin^2$ must reach $\pi/2$ so, $(P = 1)$.

$$\frac{\gamma B_0}{2\omega} = \pi/2$$

or

$$\boxed{B_0 = \frac{\pi\omega}{\gamma}}$$

Problem 5

Suppose two spin $-1/2$ particles are known to be in the singlet configuration. Let $S_a^{(1)}$ be the component of the spin angular momentum of particle number 1 in the direction defined by the unit vector $\hat{a}$. Similarly, let $S_b^{(2)}$ be the component of 2's angular momentum in the direction $\hat{b}$. Show that:

$$< S_a^{(1)} S_b^{(2)} >= -\frac{\hbar^2}{4}\cos\theta,$$

where θ is the angle between $\hat{a}$ and $\hat{b}$.

Let us chose axes so that $\hat{a}$ lies along the z-axis and $\hat{b}$ in the xz-plane. Then, $S_a^{(1)} = S_z^{(1)}$ and $S_b^{(2)} = \cos\theta S_z^{(2)} + \sin\theta S_x^{(2)}$.

Now let us calculate $\langle 00| S_a^{(1)} S_b^{(2)} |00\rangle$, where $|00\rangle = \frac{1}{\sqrt{2}} |\uparrow\downarrow - \downarrow\uparrow\rangle$.

$$\begin{aligned}
S_a^{(1)} S_b^{(2)} |00\rangle &= \frac{1}{\sqrt{2}} \left[S_z^{(1)} (\cos\theta\ S_z^{(2)} + \sin\theta\ S_x^{(2)})\right](\uparrow\downarrow - \downarrow\uparrow) \\
&= \frac{1}{\sqrt{2}} \left[(S_z \uparrow)(\cos\theta S_z \downarrow + \sin\theta\ S_x \downarrow) - (S_z \downarrow)(\cos\theta\ S_z \uparrow + \sin\theta\ S_x \uparrow)\right] \\
&= \frac{1}{\sqrt{2}} \left\{ \left(\frac{\hbar}{2} \uparrow\right) \left[\cos\theta \left(-\frac{\hbar}{2} \downarrow\right) + \sin\theta \left(\frac{\hbar}{2} \uparrow\right)\right] - \left(\frac{\hbar}{2} \downarrow\right) \left[\cos\theta \left(\frac{\hbar}{2} \uparrow\right) + \sin\theta \left(\frac{\hbar}{2} \downarrow\right)\right] \right\} \\
&= \frac{\hbar^2}{4} \left[\cos\theta \frac{1}{\sqrt{2}} (-\uparrow\downarrow + \downarrow\uparrow) + \sin\theta \frac{1}{\sqrt{2}} (\uparrow\uparrow + \downarrow\downarrow)\right] \\
&= \frac{\hbar^2}{4} \left[-\cos\theta\, |00\rangle + \sin\theta \frac{1}{\sqrt{2}} |11\rangle\right]
\end{aligned}$$

Remind that:

$$\boxed{S_x \uparrow = \frac{\hbar}{2} \downarrow}$$

$$\boxed{S_x \downarrow = \frac{\hbar}{2} \uparrow}$$

$$\boxed{S_y \uparrow = -\frac{\hbar}{2i} \downarrow}$$

$$\boxed{S_y \downarrow = \frac{\hbar}{2i} \uparrow}$$

So,

$$\begin{aligned} < S_a^{(1)} S_b^{(2)} > &= \langle 00| \, S_a^{(1)} S_b^{(2)} \, |00\rangle \\ &= \frac{\hbar^2}{4} \langle 00| \Big[-\cos\theta \, |00\rangle + \sin\theta \frac{1}{\sqrt{2}} \, |11\rangle \Big] \\ &= -\frac{\hbar^2}{4} \cos\theta \;\; \langle 00|00\rangle \end{aligned}$$

Hence,

$$\boxed{< S_a^{(1)} S_b^{(2)} > = -\frac{\hbar^2}{4} \cos\theta}$$