Physics 1

UNIVERSITÄT SIEGEN
FESTKÖRPERPHYSIK

Lecture 7b : LCR oscillators

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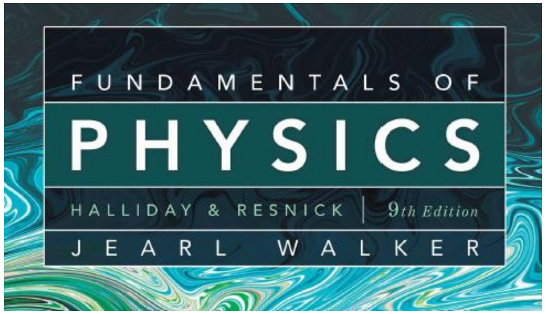
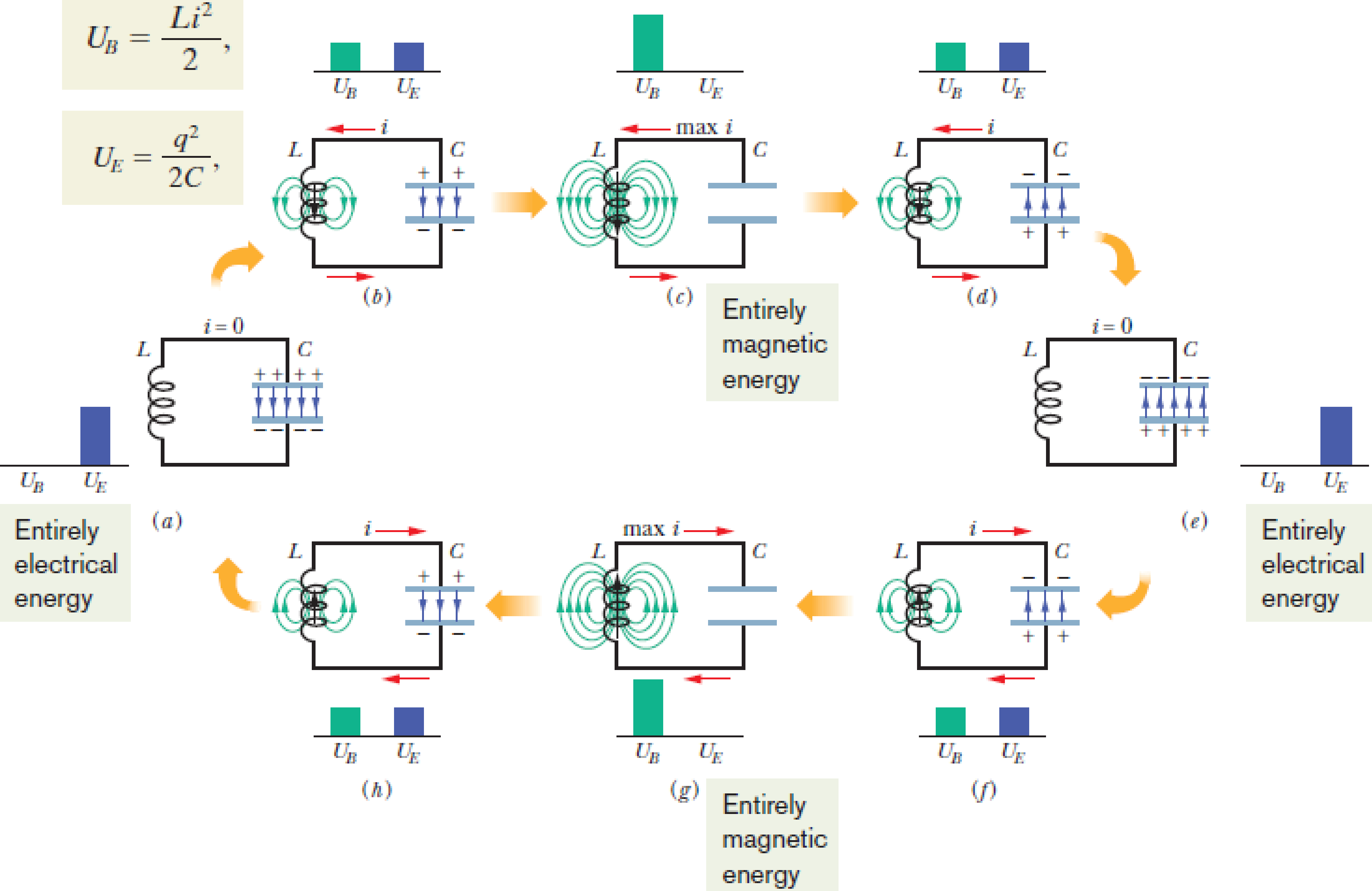


$$U_B = \frac{Li^2}{2},$$

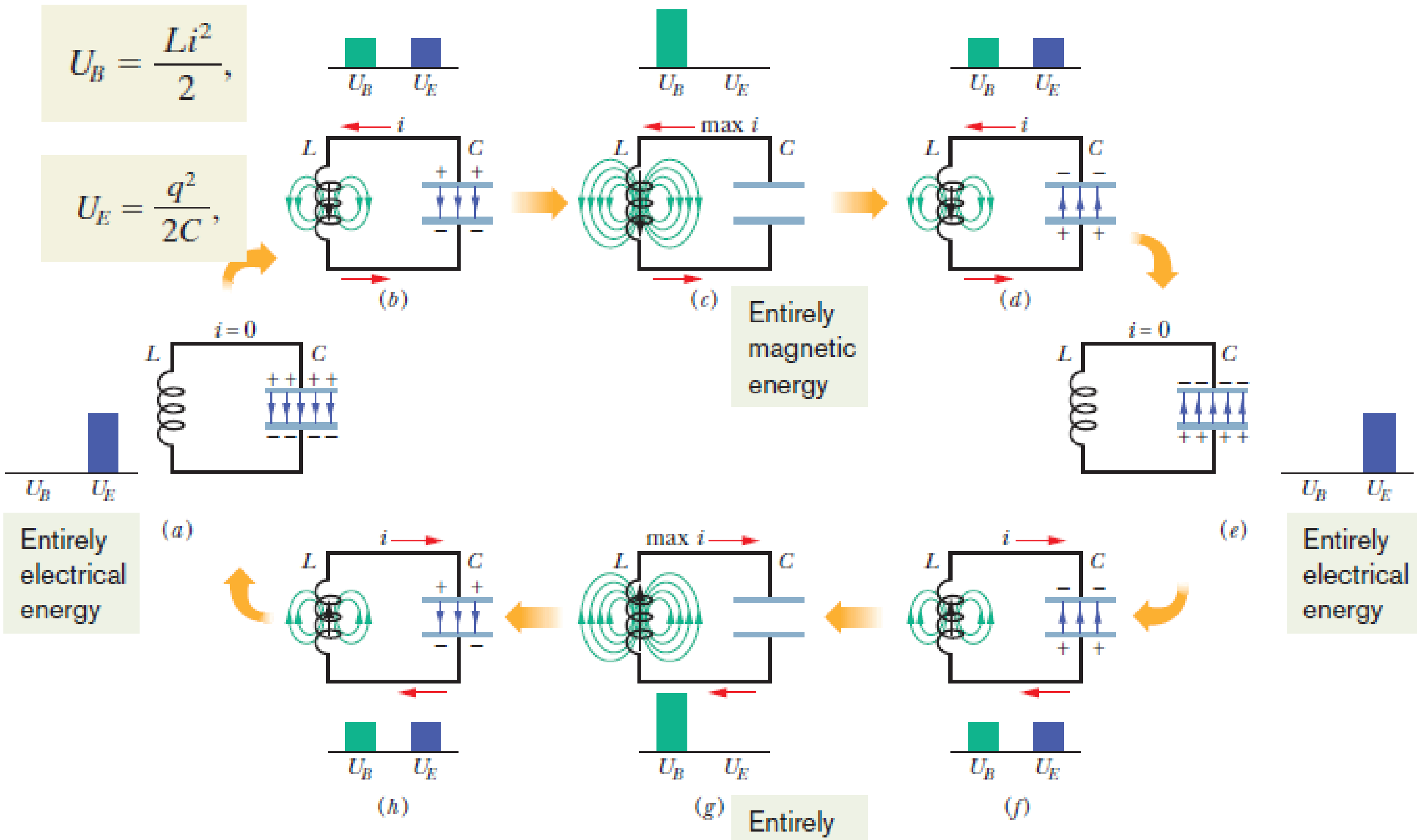
$$U_E = \frac{q^2}{2C},$$



current, and potential difference do not decay exponentially with time but vary sinusoidally (with period T and angular frequency v). Oscillations of a capacitor's electric field and the inductor's magnetic field are ***electromagnetic oscillations.***

$$U_B = \frac{Li^2}{2},$$

$$U_E = \frac{q^2}{2C},$$



(a) Capacitor with maximum charge, no current. (b) Capacitor discharging, current increasing. (c) Capacitor fully discharged, current maximum. (d) Capacitor charging but with polarity opposite that in (a), current decreasing. (e) Capacitor with maximum charge having polarity opposite that in (a), no current. (f) Capacitor discharging, current increasing with direction

Electrical - Mechanical analogy

Block–Spring System			LC Oscillator	
Element	Energy		Element	Energy
Spring	Potential, $\frac{1}{2}kx^2$		Capacitor	Electrical, $\frac{1}{2}(1/C)q^2$
Block	Kinetic, $\frac{1}{2}mv^2$		Inductor	Magnetic, $\frac{1}{2}Li^2$
	$v = dx/dt$			$i = dq/dt$

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(block–spring system).}$$

$$\omega = \frac{1}{\sqrt{LC}} \quad (LC \text{ circuit}).$$

We analyzed block–spring oscillations in Chapter 15 in terms of energy transfers and did not—at that early stage—derive the fundamental differential equation that governs those oscillations. We do so now.

We can write, for the total energy U of a block–spring oscillator at any instant,

$$U = U_b + U_s = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2, \tag{31-5}$$

where U_b and U_s are, respectively, the kinetic energy of the moving block and the potential energy of the stretched or compressed spring. If there is no friction—which we assume—the total energy U remains constant with time, even though v and x vary. In more formal language, $dU/dt = 0$. This leads to

$$\frac{dU}{dt} = \frac{d}{dt}\left(\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2\right) = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0. \tag{31-6}$$

However, $v = dx/dt$ and $dv/dt = d^2x/dt^2$. With these substitutions, Eq. 31-6 becomes

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad \text{(block–spring oscillations)}. \tag{31-7}$$

Equation 31-7 is the fundamental *differential equation* that governs the frictionless block–spring oscillations.

The general solution to Eq. 31-7—that is, the function $x(t)$ that describes the block–spring oscillations—is (as we saw in Eq. 15-3)

$$x = X\cos(\omega t + \phi) \qquad \text{(displacement)}, \tag{31-8}$$

in which X is the amplitude of the mechanical oscillations (x_m in Chapter 15), ω is the angular frequency of the oscillations, and ϕ is a phase constant.

The *LC* Oscillator

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C},$$

No loss

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Li^2}{2} + \frac{q^2}{2C}\right) = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0.$$

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \qquad \text{(}LC\text{ oscillations).}$$

$$q = Q\cos(\omega t + \phi)$$

$$i = \frac{dq}{dt} = -\omega Q\sin(\omega t + \phi)$$

$$I = \omega Q, \qquad i = -I\sin(\omega t + \phi).$$

$$\frac{d^2q}{dt^2} = -\omega^2 Q\cos(\omega t + \phi).$$

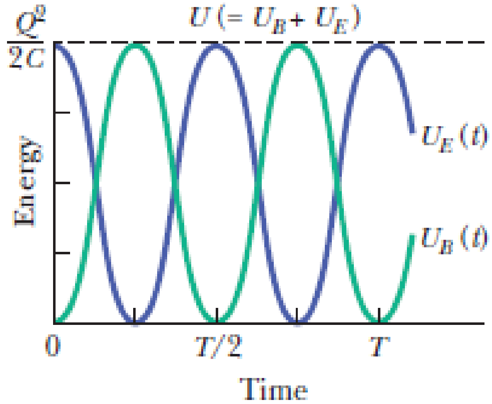
$$-L\omega^2 Q\cos(\omega t + \phi) + \frac{1}{C}Q\cos(\omega t + \phi) = 0.$$

$$\omega = \frac{1}{\sqrt{LC}}.$$

LC - oscillator

The electric energy stored in LC circuit at time t:

The electrical and magnetic energies vary but the total is constant.



(φ= 0)

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t + \phi).$$

The magnetic energy:

$$U_B = \tfrac{1}{2}Li^2 = \tfrac{1}{2}L\omega^2Q^2\sin^2(\omega t + \phi).$$

$$\omega = \frac{1}{\sqrt{LC}}$$

$$U_B = \frac{Q^2}{2C}\sin^2(\omega t + \phi).$$

1. The maximum values of U_E and U_B are both $Q^2/2C$.
2. At any instant the sum of U_E and U_B is equal to $Q^2/2C$, a constant.
3. When U_E is maximum, U_B is zero, and conversely.

LC oscillator: potential change, rate of current change

A 1.5 μF capacitor is charged to 57 V by a battery, which is then removed. At time $t = 0$, a 12 mH coil is connected in series with the capacitor to form an *LC* oscillator (Fig. 31-1).

(a) What is the potential difference $v_L(t)$ across the inductor as a function of time?

KEY IDEAS

(1) The current and potential differences of the circuit (both the potential difference of the capacitor and the potential difference of the coil) undergo sinusoidal oscillations. (2) We can still apply the loop rule to these oscillating potential differences, just as we did for the nonoscillating circuits of Chapter 27.

Calculations: At any time t during the oscillations, the loop rule and Fig. 31-1 give us

$$v_L(t) = v_C(t); \tag{31-18}$$

that is, the potential difference v_L across the inductor must always be equal to the potential difference v_C across the capacitor, so that the net potential difference around the circuit is zero. Thus, we will find $v_L(t)$ if we can find $v_C(t)$, and we can find $v_C(t)$ from $q(t)$ with Eq. 25-1 ($q = CV$).

Because the potential difference $v_C(t)$ is maximum when the oscillations begin at time $t = 0$, the charge q on the capacitor must also be maximum then. Thus, phase constant ϕ must be zero; so Eq. 31-12 gives us

$$q = Q \cos \omega t. \tag{31-19}$$

(Note that this cosine function does indeed yield maximum q ($= Q$) when $t = 0$.) To get the potential difference $v_C(t)$, we divide both sides of Eq. 31-19 by C to write

$$\frac{q}{C} = \frac{Q}{C} \cos \omega t,$$

and then use Eq. 25-1 to write

$$v_C = V_C \cos \omega t. \tag{31-20}$$

Here, V_C is the amplitude of the oscillations in the potential difference v_C across the capacitor.

Next, substituting $v_C = v_L$ from Eq. 31-18, we find

$$v_L = V_C \cos \omega t. \tag{31-21}$$

We can evaluate the right side of this equation by first noting that the amplitude V_C is equal to the initial (maximum) potential difference of 57 V across the capacitor. Then we find ω with Eq. 31-4:

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{[(0.012 \text{ H})(1.5 \times 10^{-6} \text{ F})]^{0.5}}$$

$$= 7454 \text{ rad/s} \approx 7500 \text{ rad/s}.$$

Thus, Eq. 31-21 becomes

$$v_L = (57 \text{ V}) \cos(7500 \text{ rad/s})t. \qquad \text{(Answer)}$$

(b) What is the maximum rate $(di/dt)_{\max}$ at which the current i changes in the circuit?

KEY IDEA

With the charge on the capacitor oscillating as in Eq. 31-12, the current is in the form of Eq. 31-13. Because $\phi = 0$, that equation gives us

$$i = -\omega Q \sin \omega t.$$

Calculations: Taking the derivative, we have

$$\frac{di}{dt} = \frac{d}{dt}(-\omega Q \sin \omega t) = -\omega^2 Q \cos \omega t.$$

We can simplify this equation by substituting CV_C for Q (because we know C and V_C but not Q) and $1/\sqrt{LC}$ for ω according to Eq. 31-4. We get

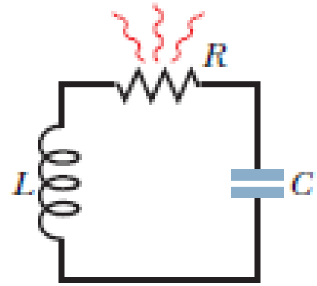
$$\frac{di}{dt} = -\frac{1}{LC} CV_C \cos \omega t = -\frac{V_C}{L} \cos \omega t.$$

This tells us that the current changes at a varying (sinusoidal) rate, with its maximum rate of change being

$$\frac{V_C}{L} = \frac{57 \text{ V}}{0.012 \text{ H}} = 4750 \text{ A/s} \approx 4800 \text{ A/s}. \qquad \text{(Answer)}$$

RCL Oscillator

As R is present, the total energy of the system is no longer constant. It decreases with time as energy is transfered to thermal energy in the resistor. Because of this, the oscillations of charge, current and potential difference continuously decrease in amplitude, and the oscillations are **damped**.



$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}.$$

The total energy at any instant

$$\frac{dU}{dt} = -i^2R,$$

The energy is decreasing as it is transferred to heat.

$$\frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2R.$$

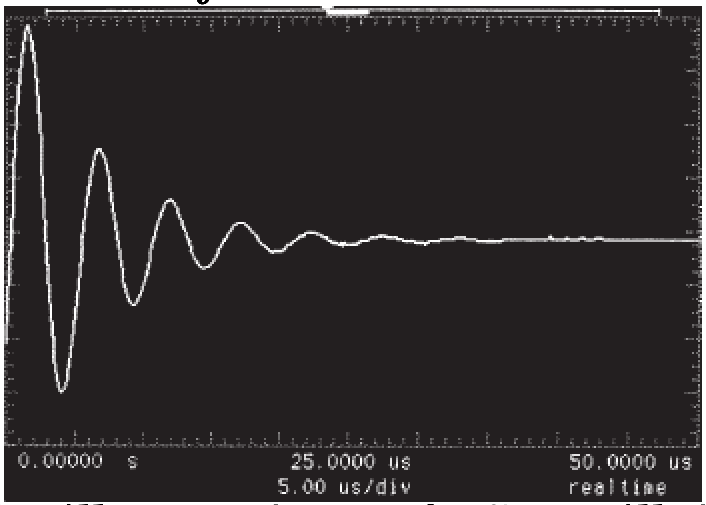
Substituting dq/dt for i and d^2q/dt^2 for di/dt,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \quad (RLC \text{ circuit}),$$

$$q = Qe^{-Rt/2L}\cos(\omega' t + \phi),$$

$$\omega' = \sqrt{\omega^2 - (R/2L)^2},$$

$$U_E = \frac{q^2}{2C} = \frac{[Qe^{-Rt/2L}\cos(\omega' t + \phi)]^2}{2C} = \frac{Q^2}{2C}e^{-Rt/L}\cos^2(\omega' t + \phi).$$



Oscilloscope trace of RCL oscillator with an **exponentially decaying amplitude**.

Damped *RLC* circuit: charge amplitude

A series *RLC* circuit has inductance $L = 12$ mH, capacitance $C = 1.6\ \mu$F, and resistance $R = 1.5\ \Omega$ and begins to oscillate at time $t = 0$.

(a) At what time t will the amplitude of the charge oscillations in the circuit be 50% of its initial value? (Note that we do not know that initial value.)

KEY IDEA

The amplitude of the charge oscillations decreases exponentially with time t: According to Eq. 31-25, the charge amplitude at any time t is $Qe^{-Rt/2L}$, in which Q is the amplitude at time $t = 0$.

Calculations: We want the time when the charge amplitude has decreased to $0.50Q$, that is, when

$$Qe^{-Rt/2L} = 0.50Q.$$

We can now cancel Q (which also means that we can answer the question without knowing the initial charge). Taking the natural logarithms of both sides (to eliminate the exponential function), we have

$$-\frac{Rt}{2L} = \ln 0.50.$$

Solving for t and then substituting given data yield

$$t = -\frac{2L}{R}\ln 0.50 = -\frac{(2)(12 \times 10^{-3}\ \text{H})(\ln 0.50)}{1.5\ \Omega}$$

$$= 0.0111\ \text{s} \approx 11\ \text{ms}. \qquad \text{(Answer)}$$

(b) How many oscillations are completed within this time?

KEY IDEA

The time for one complete oscillation is the period $T = 2\pi/\omega$, where the angular frequency for LC oscillations is given by Eq. 31-4 ($\omega = 1/\sqrt{LC}$).

Calculation: In the time interval $\Delta t = 0.0111$ s, the number of complete oscillations is

$$\frac{\Delta t}{T} = \frac{\Delta t}{2\pi\sqrt{LC}}$$

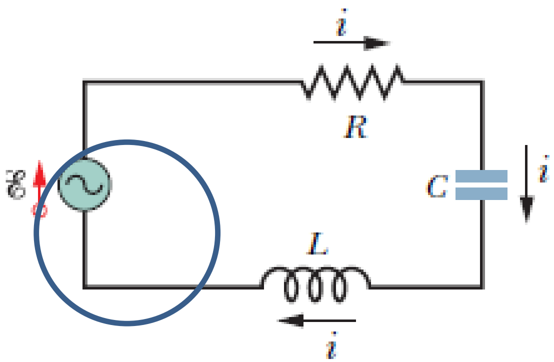
$$= \frac{0.0111\ \text{s}}{2\pi[(12 \times 10^{-3}\ \text{H})(1.6 \times 10^{-6}\ \text{F})]^{1/2}} \approx 13. \qquad \text{(Answer)}$$

Thus, the amplitude decays by 50% in about 13 complete oscillations. This damping is less severe than that shown in Fig. 31-3, where the amplitude decays by a little more than 50% in one oscillation.

Alternating Current

Emf and the current is varying sinusoidally with time, reversing direction , which means oscillating.

Forced Oscillations

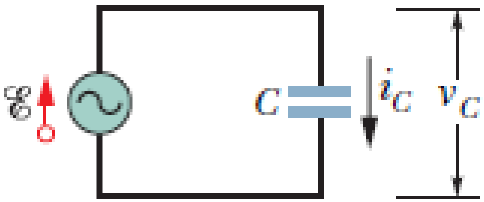


A generator, represented by a sin wave in a circle, produces an alternating emf that establishes an alternating current

When the external alternating emf is connected to an RLC circuit, the oscillations of charge, potential difference, and current are driven oscillations or forced oscillations. These oscillations always occur at the driving angular frequency ω_d.

When eigenfrequeny ω equals the driving frequqncy ω_d , **resonance** *occurs, the amplitude of I is maximum.*

A capacitive load



$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t. \qquad i = I \sin(\omega_d t - \phi),$$

$$\varepsilon - Vc = 0$$

$$v_C = V_C \sin \omega_d t,$$

$$q_C = C v_C = C V_C \sin \omega_d t.$$

$$i_C = \frac{dq_C}{dt} = \omega_d C V_C \cos \omega_d t.$$

$$X_C = \frac{1}{\omega_d C} \qquad \text{(capacitive reactance).}$$

Xc has unit ohm, just as R

$$\cos \omega_d t = \sin(\omega_d t + 90°).$$

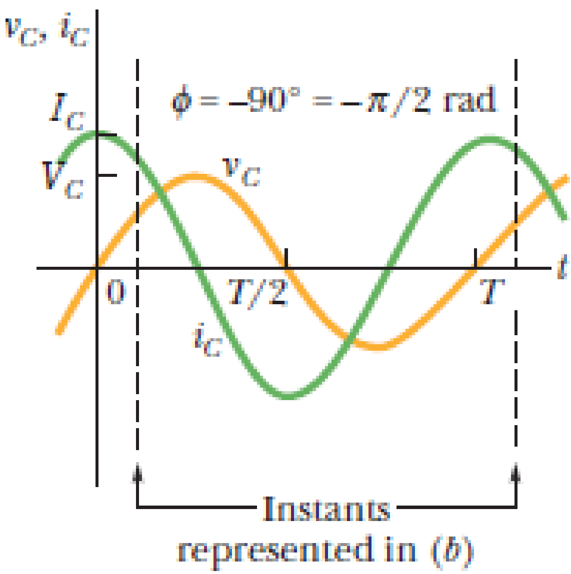
$$i_C = \left(\frac{V_C}{X_C}\right) \sin(\omega_d t + 90°).$$

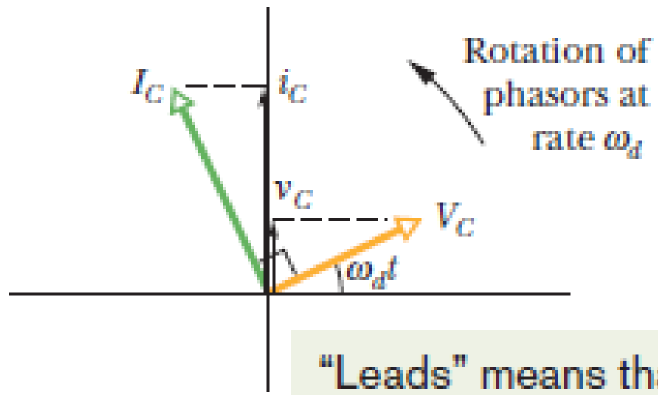
$$i_C = I_C \sin(\omega_d t - \phi),$$

For a purely capacitive load the phase constant φ for the current is -90° with respect to applied EMF.

$$V_C = I_C X_C \qquad \text{(capacitor).}$$

For a capacitive load, the current leads the potential difference by 90°.

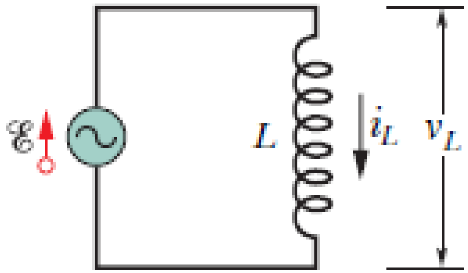




"Leads" means that the current peaks at an *earlier* time than the potential difference.

The quantities v_C and i_C are 90°, π/2 rad, or one-quarter cycle, out of phase. Furthermore, we see that iC leads vC, which means that, if you monitored the current iC and the potential difference vC in the circuit, you would find that iC reaches its maximum before vC does, by one-quarter cycle.
The phasors representing these two quantities rotate counterclockwise together, the phasor labeled IC does indeed lead that labeled VC, and by an angle of 90°; that is, the phasor IC coincides with the vertical axis one-quarter cycle before the phasor VC does.

An inductive load



$$v_L = V_L \sin \omega_d t,$$

$$v_L = L \frac{di_L}{dt}.$$

$$\frac{di_L}{dt} = \frac{V_L}{L} \sin \omega_d t.$$

$$i_L = \int di_L = \frac{V_L}{L} \int \sin \omega_d t \, dt = -\left(\frac{V_L}{\omega_d L}\right) \cos \omega_d t.$$

$$X_L = \omega_d L \qquad \text{(inductive reactance).}$$

$$-\cos \omega_d t = \sin(\omega_d t - 90°).$$

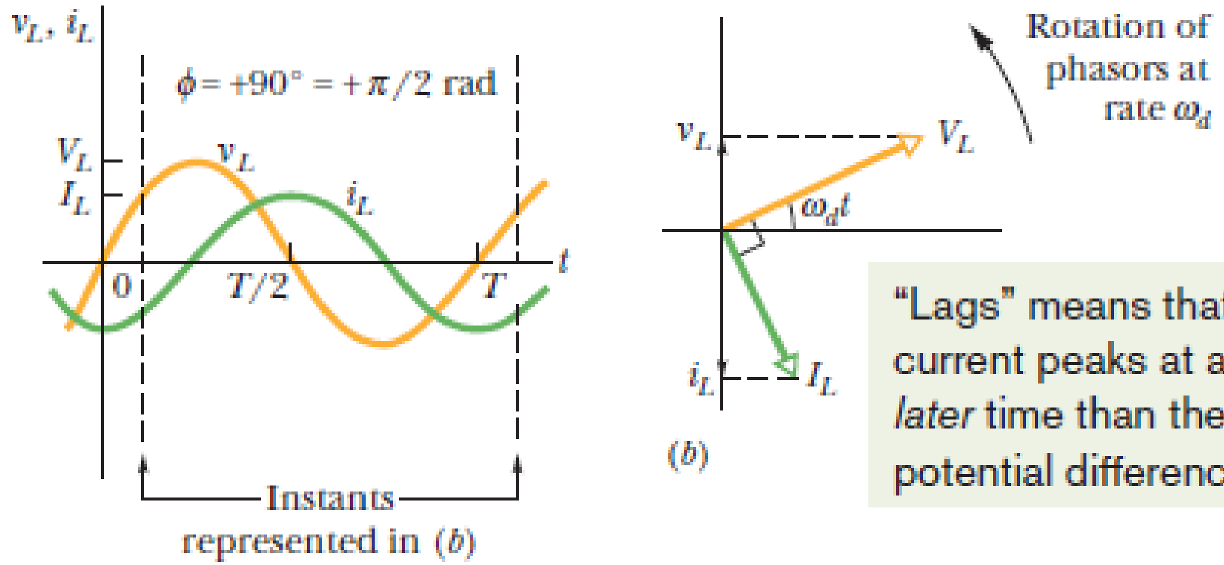
$$i_L = \left(\frac{V_L}{X_L}\right) \sin(\omega_d t - 90°).$$

$$i_L = I_L \sin(\omega_d t - \phi),$$

a purely inductive load the phase constant f for the current is 90° with respect to applied EMF

$$V_L = I_L X_L \qquad \text{(inductor)}.$$

For an inductive load, the current lags the potential difference by 90°.



"Lags" means that the current peaks at a *later* time than the potential difference.

the quantities iL and vL are 90° out of phase. In this case, however, iL lags vL; that is, monitoring the current iL and the potential difference vL in the circuit shows that iL reaches its maximum value after vL does, by one-quarter cycle.

The phasor diagram also contains this information. As the phasors rotate counterclockwise in the figure, the phasor labeled IL does indeed lag that labeled VL, and by an angle of 90°.

Phase and Amplitude Relations for Alternating Currents and Voltages

Circuit Element	Symbol	Resistance or Reactance	Phase of the Current	Phase Constant (or Angle) ϕ	Amplitude Relation
Resistor	R	R	In phase with v_R	$0°$ ($= 0$ rad)	$V_R = I_R R$
Capacitor	C	$X_C = 1/\omega_d C$	Leads v_C by $90°$ ($= \pi/2$ rad)	$-90°$ ($= -\pi/2$ rad)	$V_C = I_C X_C$
Inductor	L	$X_L = \omega_d L$	Lags v_L by $90°$ ($= \pi/2$ rad)	$+90°$ ($= +\pi/2$ rad)	$V_L = I_L X_L$

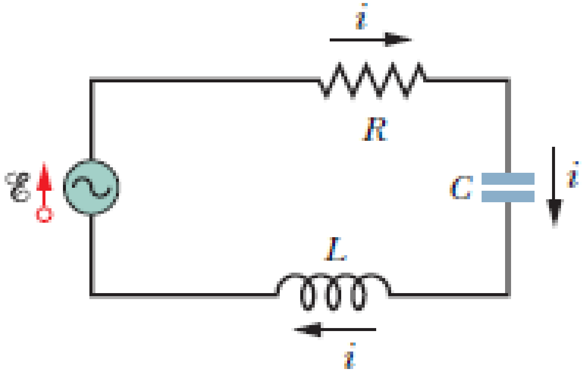
When an applied alternating voltage produces an alternating current in these elements, the current is always in phase with the voltage across a resistor, always leads the voltage across a capacitor, and always lags the voltage across an inductor.

RLC circuit

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \qquad \text{(impedance defined).}$$

$$I = \frac{\mathscr{E}_m}{Z}.$$

Complex Ohm´s law



$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \qquad \text{(current amplitude).}$$

The current that we have been describing in this section is the steady-state current that occurs after the alternating emf has been applied for some time. When the emf is first applied to a circuit, a brief transient current occurs. Its duration is determined by the time constants $\tau_L = L/R$ and $\tau_c = RC$ as the inductive and capacitive elements "turn on."

$$\tan \phi = \frac{X_L - X_C}{R} \qquad \text{(phase constant).}$$

$$\tan \phi = \frac{X_L - X_C}{R} \qquad \text{(phase constant).}$$

***$X_L > X_C$**: The circuit is more inductive than capacitive*
***$X_C > X_L$**: The circuit is more capacitive than inductive.*
***$X_C = X_L$**: The circuit is in resonance*

the current amplitude I in an RLC circuit

$$I = \frac{\mathcal{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}}$$

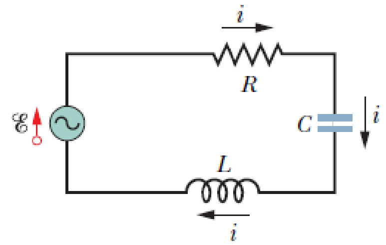
For a given resistance R, that amplitude is a maximum when the quantity

$\omega_d L - 1/\omega_d C = 0$

the natural angular frequency of the RLC circuit

$$\omega_d = \frac{1}{\sqrt{LC}} \qquad \text{(maximum } I\text{).}$$

$$\omega_d = \omega = \frac{1}{\sqrt{LC}} \qquad \text{(resonance).}$$



Current amplitude, impedance, and phase constant

In Fig. 31-7, let $R = 200\ \Omega$, $C = 15.0\ \mu F$, $L = 230$ mH, $f_d = 60.0$ Hz, and $\mathscr{E}_m = 36.0$ V. (These parameters are those used in the earlier sample problems above.)

(a) What is the current amplitude I?

KEY IDEA

The current amplitude I depends on the amplitude $\mathscr{E}_m$ of the driving emf and on the impedance Z of the circuit, according to Eq. 31-62 ($I = \mathscr{E}_m/Z$).

Calculations: So, we need to find Z, which depends on resistance R, capacitive reactance X_C, and inductive reactance X_L. The circuit's resistance is the given resistance R. Its capacitive reactance is due to the given capacitance and, from an earlier sample problem, $X_C = 177\ \Omega$. Its inductive reactance is due to the given inductance and, from another sample problem, $X_L = 86.7\ \Omega$. Thus, the circuit's impedance is

$$\begin{aligned} Z &= \sqrt{R^2 + (X_L - X_C)^2} \\ &= \sqrt{(200\ \Omega)^2 + (86.7\ \Omega - 177\ \Omega)^2} \\ &= 219\ \Omega. \end{aligned}$$

We then find

$$I = \frac{\mathscr{E}_m}{Z} = \frac{36.0\ \text{V}}{219\ \Omega} = 0.164\ \text{A}. \qquad \text{(Answer)}$$

(b) What is the phase constant ϕ of the current in the circuit relative to the driving emf?

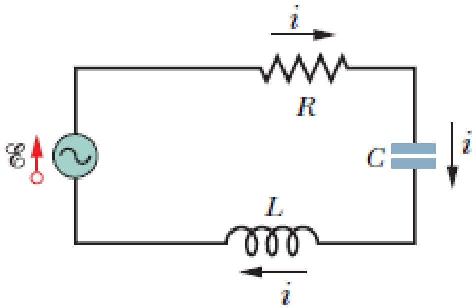
KEY IDEA

The phase constant depends on the inductive reactance, the capacitive reactance, and the resistance of the circuit, according to Eq. 31-65.

Calculation: Solving Eq. 31-65 for ϕ leads to

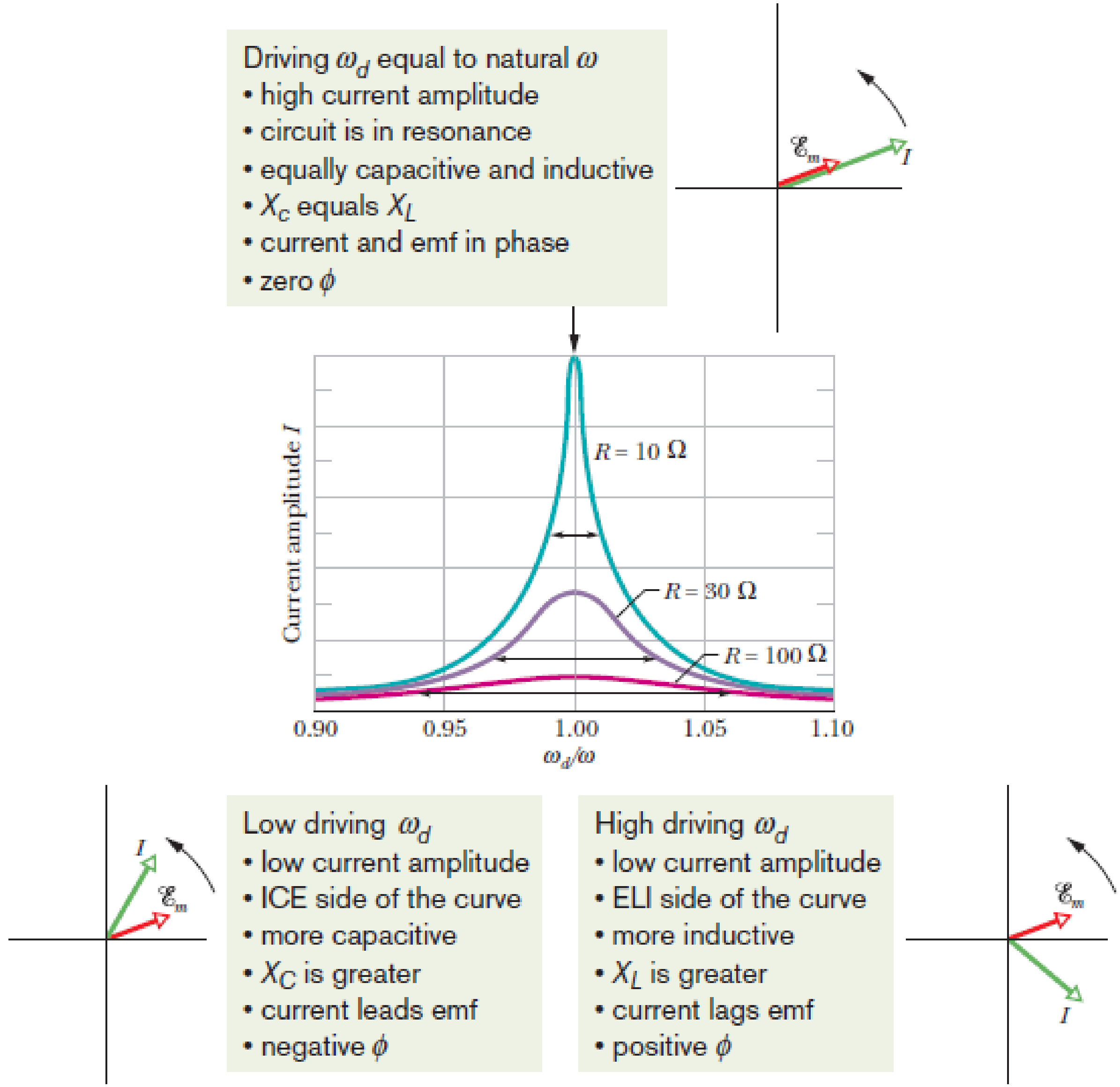
$$\begin{aligned} \phi &= \tan^{-1}\frac{X_L - X_C}{R} = \tan^{-1}\frac{86.7\ \Omega - 177\ \Omega}{200\ \Omega} \\ &= -24.3° = -0.424\ \text{rad}. \qquad \text{(Answer)} \end{aligned}$$

The negative phase constant is consistent with the fact that the load is mainly capacitive; that is, $X_C > X_L$. In the common mnemonic for driven series RLC circuits, this circuit is an *ICE* circuit—the current *leads* the driving emf.



Each curve peaks at its maximum current amplitude I when the ratio ω_d / ω is 1, but the maximum value of I decreases with increasing R. In addition, the curves increase in width with increasing R

R- damping term.



$$V(t) - V_C - V_L - V_R = V(t) - \frac{Q}{C} - L\frac{dI}{dt} - IR = 0$$

using $V(t) = V_0 e^{i\omega t}$

$$L\frac{dI}{dt} + IR + \frac{q}{C} = V_0 e^{i\omega t}$$

$$L\frac{d^2I}{dt^2} + \frac{dI}{dt}R + \frac{I}{C} = i\omega V_0 e^{i\omega t}$$

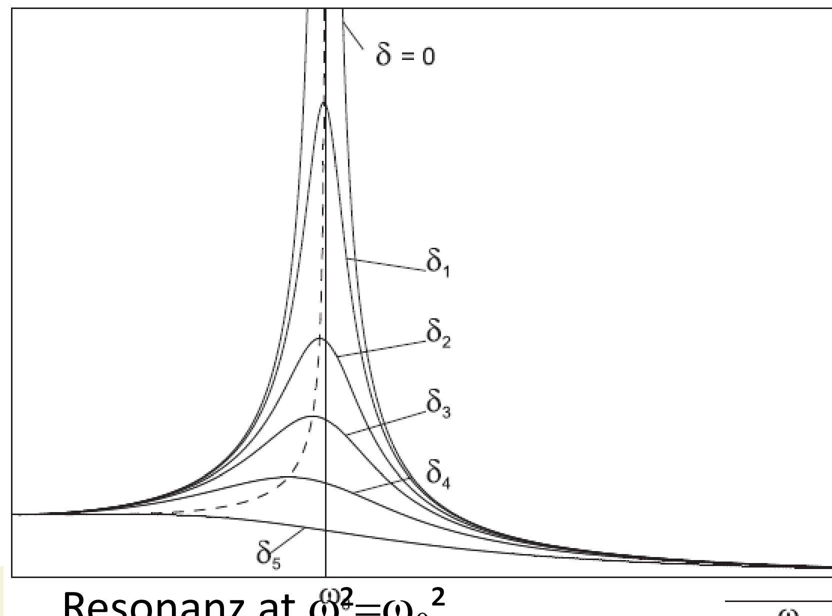
Ansatz: $I = I_0 e^{i\omega t}$

$$-I_0 L\omega^2 e^{i\omega t} + i\omega I_0 R e^{i\omega t} + I_0 e^{i\omega t}/C = i\omega V_0 e^{i\omega t}$$

$$-I_0\omega^2 + i\omega I_0 \frac{R}{L} + \frac{I_0}{LC} = i\omega V_0/L$$

$$I_0((\omega_0^2 - \omega^2) + i\omega\frac{R}{L}) = i\omega V_0/L$$

$$I_0 = \frac{i\omega V_0/L}{((\omega^2 - \omega_0^2) + i\omega\frac{R}{L})} = \frac{i\omega V_0/L}{\sqrt{((\omega^2 - \omega_0^2)^2 - (\omega\frac{R}{L})^2}}$$



$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \quad \text{(current amplitude).}$$

Resonanz at $\omega^2 = \omega_0^2$

$I \rightarrow \infty$ for $R/L \rightarrow 0$