Theory of Quantum Matter

Lecturer: Prof. Otfried Gühne (Mon 14:00, Fri 10:00, Room D120)
Exercises: Chau Nguyen (Fri 12:30, Room D201) **Sheet 12**

Hand in: not applied
Discussion date: not applied (questions can be discussed on 18.01.2019)

19. The Cooper problem

Here we explore some details of the Cooper problem: two electrons outside the Fermi ball tend to form a bound state under *arbitrary* weak attraction. Recall that the wave function for the two electrons in momentum space $\varphi_{\vec{k}}$ is the solution of the Schrödinger equation,

$$2\epsilon_{\vec{k}}\varphi_{\vec{k}} - V\sum_{\vec{k}'}\varphi_{\vec{k}'} = E\varphi_{\vec{k}}, \tag{1}$$

where $E_F - \hbar\omega_D \leq \epsilon_{\vec{k}'} \leq E_F + \hbar\omega_D$. Here $\epsilon_{\vec{k}}$ is the electron dispersion relation, ω_D is the Debye frequency, E_F is the Fermi energy, and V is the strength of the attraction. This ultimately leads to the equation for the energy,

$$1 = \sum_{\vec{k}} \frac{V}{2\epsilon_{\vec{k}} - E}, \tag{2}$$

where the summation is limited to $E_F \leq \epsilon_{\vec{k}} \leq E_F + \hbar\omega_D$. Transforming this to integral, we have

$$1 = V\int_{E_F}^{E_F+\hbar\omega_D} \mathrm{d}\,\epsilon \frac{\rho(\epsilon)}{2\epsilon - E}. \tag{3}$$

where $\rho(\epsilon)$ is the electron density of state. The solution for the case $\hbar\omega_D \ll E_F$ where the slow-varying function $\rho(\epsilon)$ can be treated as constant has been given explicitly in the lectures.

(a) Without invoking the above approximation, provided $E_F > 0$, show that equation (3) *always* has a bound state solution with binding energy $\Delta = E - 2E_F < 0$ for arbitrary weak V. To appreciate the importance of the presence of the Fermi surface: show that if $E_F = 0$ (i.e., two electrons in the vacuum), for sufficiently small V, the equation has no bound state solution.

Hint: Study the monotonicity of the right hand side of equation (3) for $E < 2E_F$.

Let us now estimate the size of the Cooper pair. Recall that the wave function in the momentum space upto a normalisation factor is given by

$$\varphi_{\vec{k}} \propto \frac{1}{2\epsilon_{\vec{k}} - E}, \tag{4}$$

for $\epsilon_{\vec{k}} \geq E_F$. The wave function in real space can be found by Fourier transformation,

$$\varphi(\vec{r}) = \sum_{\vec{k}} \varphi_{\vec{k}} e^{i\vec{k}\vec{r}}, \tag{5}$$

where $\vec{r}$ is the difference in the coordinates of the two electrons. The mean square radius of such a state is

$$R^2 = \frac{\int \mathrm{d}\,\vec{r}|\varphi(\vec{r})|^2 r^2}{\int \mathrm{d}\,\vec{r}|\varphi(\vec{r})|^2}. \tag{6}$$

(b) Show that one can also write

$$R^2 = \frac{\int \mathrm{d}\,\vec{k} |\nabla_{\vec{k}} \varphi_{\vec{k}}|^2}{\int \mathrm{d}\,\vec{k} |\varphi_{\vec{k}}|^2}. \tag{7}$$

(c) Using the approximation as in the lecture ($\Delta \ll \hbar\omega_D \ll E_F$; slow-varying functions under integrals are treated as constants, etc.) to evaluate the integral to show that

$$R \approx \frac{2}{\sqrt{3}} \frac{\hbar v_F}{\Delta}, \tag{8}$$

where v_F is the Fermi velocity (corresponding to the energy E_F).

Remark: With realistic parameters, one finds that $R \approx 1\mu m$, which is way above the atomic scale. One cannot simply consider Cooper pairs as single particles.