Quantum Theory of Light (WS17/18)

Exercises 3

(For exercise class on Thu, January 11th 2018.)

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1. **Partial trace.**
   (a) Prove, that the eigenvalues of the reduced density matrix $\mathrm{tr}_B |\psi\rangle\langle\psi|$ of an arbitrary bipartite state $|\psi\rangle = \sum_{ij} \Psi_{ij} |ij\rangle$ are the squares of the Schmidt coefficients of $|\psi\rangle$, i.e., the singular values of matrix Ψ.
   (b) Prove, that a purification of a density matrix is not entangled if and only if the density matrix represents a pure state. (Hint: A purification of ρ obeys $\rho = \mathrm{tr}_{\mathrm{aux}} |\psi\rangle\langle\psi|$.)

2. **Dilations and embeddings.**
   (a) Consider a family of positive semidefinite operators $(E_1, \dots, E_n)$ on a Hilbert space $\mathcal{H}$ with $\sum_k E_k = \mathbb{1}_{\mathcal{H}}$. Show that there exists some projections $(\Pi_1, \dots, \Pi_n)$ on an appropriate Hilbert space $\mathcal{H}'$ with $\sum_k \Pi_k = \mathbb{1}_{\mathcal{H}'}$, such that $E_k = Q^\dagger \Pi_k Q$, where $Q\colon \mathcal{H} \to \mathcal{H}'$ with a linear map obeying $Q^\dagger Q = \mathbb{1}_{\mathcal{H}}$. (Hints: First show, that it is sufficient to consider the case where $E_k = |e_k\rangle\langle e_k|$ with unnormalized vectors $|e_k\rangle$. The matrix of Q simply is a $\dim(\mathcal{H}') \times \dim(\mathcal{H})$ matrix in which the first $\dim(\mathcal{H})$ rows are the identity matrix and the remaining ones are all 0's.)
   (b) Assume, Λ is a completely positive map. Show, that
   $$\chi = \sum_{ij} (\Lambda \otimes \mathrm{id})(|i\rangle\langle j| \otimes |i\rangle\langle j|)$$
   is a positive semi-definite matrix, where $\mathrm{id}\colon X \mapsto X$ is the identity map.
   (c) Conversely, show, that if χ' is a positive semi-definite matrix, then there exist operators A_k, such that
   $$\mathrm{tr}_A\{\chi'(\rho^T \otimes \mathbb{1})\} = \sum A_k \rho A_k^\dagger,$$
   where $\rho^T = \sum_{ij} |i\rangle\langle j| \langle j|\rho|i\rangle$ is the transpose without conjugation. (Hint: Use the spectral decomposition of χ'.)

3. **The Clauser–Horne–Shimony–Holt inequality.**
   Two parties, Alice and Bob, measure a shared (quantum) system. Each of them randomly chooses between two observables 0 and 1, where either of the observables have only outcomes ± 1. We define
   $$S = E(0,0) + E(0,1) + E(1,0) - E(1,1),$$
   where $E(x,y)$ is the product of the outcomes of Alice and Bob, if the observables x and y have been measured, respectively.
   (a) Under the assumption that the four observables have fixed values assigned to them, i.e., $E(x,y) = A_x B_y$ with $A_x \in \{+1, -1\}$ and $B_y \in \{+1, -1\}$, compute the maximal value s that S can attain.
   (b) Assume now, that the joint system is described by a global classical variable λ and λ follows the probability distribution $p(\lambda)\mathrm{d}\lambda$. How does the maximal value of the average $\langle S\rangle \equiv \int S(\lambda)p(\lambda)\mathrm{d}\lambda$ change?
   (c) In quantum theory, $\langle E(x,y)\rangle = \mathrm{tr}\{\rho(\hat{A}_x \otimes \hat{B}_y)\}$. Assume, that the observables $\hat{A}_0$ and $\hat{B}_0$ are represented by σ_x and the observables $\hat{A}_1$ and $\hat{B}_1$ are represented by σ_z. Find a quantum state ρ with $\langle S\rangle = 2\sqrt{2}$.
   (d) Connect the insights so far to the statements of Bells theorem.
   (e) Show, that according to quantum theory, $\langle S\rangle > 2\sqrt{2}$ is not possible. (Hint: Calculate S_{qm}^2, where $E_{\mathrm{qm}}(x,y) = \hat{A}_x \otimes \hat{B}_y$.)